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STABILITY OF NONLINEAR SYSTEMS WITH MULTI-TERM HILFER FRACTIONAL DERIVATIVES AND VARIABLE COEFFICIENTS

H. WANG[1], F. LI[1*]

Abstract. This paper investigates the stability of nonlinear system with multi-term Hilfer fractional derivatives and variable coefficients, which is a challenging problem. We establish the existence and uniqueness of solutions for the given system, as well as its Ulam-Hyers and Ulam-Hyers-Mittag-Leffler stability in a weighted space. To validate our theoretical findings, we present three representative examples, among which an RLC circuit model is analyzed in detail.

Keywords: fractional nonlinear system, variable coefficient, Ulam-Hyers-Mittag-Leffler stability, fractional-order RLC circuits, Brownian motion.

AMS Subject Classification: 26A33, 34A08, 34A34, 93D05.

1. Introduction

As a generalization of classical integer-order operators, fractional calculus provides the mathematical capabilities for describing hereditary processes and memory-dependent behaviors. These characteristics make fractional-order models particularly valuable for analyzing dynamical systems with non-local temporal dependencies. The governing equations of such systems, known as fractional differential equations (FDEs), have become indispensable in multidisciplinary research, demonstrating remarkable effectiveness in domains ranging from sciences to engineering [2, 5-7, 14, 20, 21, 23, 24, 27, 28, 30, 34, 37, 40], with applications spanning circuits, population dynamics, neural networks, robotics, and control systems. In 1908, Langevin established the mathematical model for Brownian motion through a stochastic differential equation. Subsequently, this equation was extended by Mainardi and Pironi via fractional-order operators to describe anomalous diffusion in complex systems, resulting in the fractional Langevin equations (FLEs). Significant research has focused on investigating the initial/boundary value problems of FLEs, as demonstrated in references [8, 9, 10, 13, 42]. The inclusion of non-constant coefficient functions introduces significant challenges in deriving solution representations for variable-coefficient FLEs, particularly in nonlinear cases. Recent studies [29, 32, 33] have made some progress in addressing the linear cases.

The foundational concept of Ulam-Hyers (UH) stability was first introduced in 1940 by Ulam and Hyers [16, 36]. This seminal result guarantees that for any equation satisfying UH stability criteria, an exact solution necessarily exists within a rigorously defined neighborhood of its approximate solution.

Rassias (1978) generalized the Ulam stability concept by incorporating functional variables, establishing the extended Ulam-Hyers-Rassias (UHR) stability framework [31]. Moreover, when the functional variable takes the specific form of a Mittag-Leffler function, the stability is classified as Ulam-Hyers-Mittag-Leffler (UHML) stability (see [4, 38, 39]). However, investigations into UHML stability for such equations remain relatively scarce in the existing literature.

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[1] School of Mathematics, Yunnan Normal University, Kunming, 650500, P.R. China
[*] Corresponding author: fangli860@ynnu.edu.cn

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Fractional-order systems (FOSs) serve as a versatile modeling and control framework with broad applications, including: Chaos theory [43], fractional dynamics, financial economics [12], and complex physical processes [3]. Recent advances have spurred significant research interest in FOSs [1, 3, 11, 17, 22, 26, 35] and references therein.

In [3], the authors comprehensively analyzed boundary value problems for diverse classes of FOSs, incorporating different fractional derivatives. In [26], the existence and uniqueness of solutions were established for the following ψ-Hilfer FOS with boundary value conditions:

$$\begin{aligned} {}^H D_{a^+}^{\alpha_1,\beta;\psi} x(t) &= f(t, y(t), {}^H D_{a^+}^{u,v;\psi} y(t)), \quad t \in [a,b], \\ {}^H D_{a^+}^{\alpha_2,\beta;\psi} y(t) &= g(t, x(t), {}^H D_{a^+}^{u,v;\psi} x(t)), \quad t \in [a,b], \end{aligned}$$

where ${}^H D^{\cdot;\psi}$ represents the ψ-Hilfer fractional derivative.

In [17], the authors demonstrated the existence and uniqueness of solutions for nonlinear Caputo-type fractional integro-differential Langevin equations under boundary value conditions

$$\begin{aligned} {}^c D^{\beta_1}({}^c D^{\alpha_1} + \lambda_1(t))x(t) &= f(t, x(t), y(t), \int_0^t \phi(t,s)y(s)ds), \quad t \in [0,1], \\ {}^c D^{\beta_2}({}^c D^{\alpha_2} + \lambda_2(t))y(t) &= g(t, x(t), y(t), \int_0^t \psi(t,s)x(s)ds), \quad t \in [0,1], \end{aligned}$$

where $0 < \alpha_k < 1, 1 < \beta_k \leq 2$, for $k = 1,2$, $\lambda_1, \lambda_2 : [0,1] \to \mathbb{R}$ are continuous functions, the functions $\phi, \psi : [0,1] \times [0,1] \to [0, +\infty)$.

However, to the best of our knowledge, research on Hilfer fractional systems with variable coefficients remains limited. In this study, we conduct a comprehensive investigation and analysis of UH and UHML stability for fractional Langevin systems. Specifically, we examine the following nonlinear multi-term Hilfer fractional system with variable coefficients:

$$\begin{cases} \chi_1(D)p(t) = g_1(t, q(t), {}^H D_{0^+}^{\alpha_1,\beta_1} q(t)), \ t \in J, \\ \chi_2(D)q(t) = g_2(t, p(t), {}^H D_{0^+}^{\alpha_1,\beta_1} p(t)), \ t \in J, \\ (I_{0^+}^{1-\gamma_1} p)(0^+) = p_0, \quad (I_{0^+}^{1-\gamma_1} q)(0^+) = q_0, \end{cases} \tag{1}$$

where $0 < \alpha_2 \leq \gamma_2 < \alpha_1 \leq \gamma_1 < 1$, $0 \leq \beta_i < 1$, $\gamma_i = \alpha_i + \beta_i(1-\alpha_i)\,(i = 1,2)$, $J := (0,T]$,

$$\begin{aligned} \chi_1(D) &= \left[{}^H D_{0^+}^{\alpha_2,\beta_2} + \lambda_2(t)\right]\left[{}^H D_{0^+}^{\alpha_1,\beta_1} + \lambda_1(t)\right], \\ \chi_2(D) &= \left[{}^H D_{0^+}^{\alpha_2,\beta_2} + \delta_2(t)\right]\left[{}^H D_{0^+}^{\alpha_1,\beta_1} + \delta_1(t)\right], \end{aligned}$$

and λ_i, $\delta_i\,(i = 1,2)$ are continuous functions.

When $\lambda_2(t) = \delta_2(t) = 0$, (1) reduces to a fractional Langevin system with variable coefficients; when $\lambda_1(t) = \delta_1(t) = 0$, $\chi_1(D)$ and $\chi_2(D)$ denote the composite of Hilfer fractional derivative operators, the corresponding system constitutes a mathematical framework for describing an RLC circuit (see [44]).

To the best of our knowledge, no prior studies in the relevant literature have addressed the existence, uniqueness of solutions for the system given in (1), as well as UH stability and UHML stability of (1).

The analysis of the aforementioned system presents several fundamental challenges:

(i) the inclusion of variable coefficients significantly complicates the derivation of representations of solutions;
(ii) the inherent complexity of the above system renders it unsolvable through direct integration methods;
(iii) the presence of multiple derivatives introduces additional difficulties in the stability analysis.

Consequently, we shift our focus to the following system:

$$\begin{cases} {}^{H}D_{0^+}^{\alpha_2,\beta_2}[\,{}^{H}D_{0^+}^{\alpha_1,\beta_1} + \lambda_1(t)]p(t) = f_1(t, p(t),\ {}^{H}D_{0^+}^{\alpha_1,\beta_1}p(t), q(t),\ {}^{H}D_{0^+}^{\alpha_1,\beta_1}q(t)), \\ {}^{H}D_{0^+}^{\alpha_2,\beta_2}[\,{}^{H}D_{0^+}^{\alpha_1,\beta_1} + \delta_1(t)]q(t) = f_2(t, p(t),\ {}^{H}D_{0^+}^{\alpha_1,\beta_1}p(t), q(t),\ {}^{H}D_{0^+}^{\alpha_1,\beta_1}q(t)), \\ (I_{0^+}^{1-\gamma_1}p)(0^+) = p_0, \quad (I_{0^+}^{1-\gamma_1}q)(0^+) = q_0. \end{cases} \tag{2}$$

We establish the existence and uniqueness of solutions for the system (2), as well as UH stability and UHML stability of the system (2). As a corollary of our analysis, we obtain new results for (1) that remain novel even in the case of constant coefficients.

Our primary methodological and theoretical advances include:

(i) we obtain the existence and uniqueness of solutions for the nonlinear Langevin system with variable coefficients in a weighted space;
(ii) we provide an explicit representation of the solution for the linear system corresponding to the nonlinear system;
(iii) we establish the stability conditions for the nonlinear Langevin system with variable coefficients in a weighted space;
(iv) the theoretical framework is successfully applied to analyze a fractional-order RLC circuit.

This study is organized into seven sections. Section 2 presents essential definitions and properties of fractional derivatives. Section 3 is devoted to investigating the existence and uniqueness of solutions for (2). In Section 4, we analyze the stability of (2). Section 5 derives the corresponding results for the system (1). Section 6 demonstrates our findings through three examples. Finally, Section 7 concludes with a summary of the paper.

2. PRELIMINARIES

Let $0 \leq t_0 < T$, $C[t_0, T]$ indicates the Banach space of continuous functions z on $[t_0, T]$ with the maximum norm $\|z\|_C = \max\limits_{t\in[t_0,T]} |z(t)|$. For $0 \leq \sigma < 1$, $C_\sigma[t_0, T]$ represents the weighted space $C_\sigma[t_0, T] = \{z \in C(t_0, T];\ (t - t_0)^\sigma z \in C[t_0, T]\}$ with the norm $\|z\|_{C_\sigma} = \max\limits_{t\in[t_0,T]} |(t - t_0)^\sigma z(t)|$. $C^1_\sigma[t_0, T]$ denotes the weighted space $C^1_\sigma[t_0, T] = \{z \in C[t_0, T];\ z' \in C_\sigma[t_0, T]\}$ with the norm $\|z\|_{C^1_\sigma} = \|z\|_C + \|z'\|_{C_\sigma}$. $C^{\alpha,\beta}_{1-\gamma}[t_0, T]$ stands for the weighted space

$$C^{\alpha,\beta}_{1-\gamma}[t_0, T] = \{z \in C_{1-\gamma}[t_0, T];\ {}^{H}D^{\alpha,\beta}_{t_0^+} z \in C_{1-\gamma}[t_0, T]\} \subset C_{1-\gamma}[t_0, T],\ \gamma = \alpha + \beta(1 - \alpha),$$

with the norm $\|z\|_{C^{\alpha,\beta}_{1-\gamma}} = \|z\|_{C_{1-\gamma}} + \|{}^{H}D^{\alpha,\beta}_{t_0^+} z\|_{C_{1-\gamma}}$.

Clearly, $C_0[t_0, T] = C[t_0, T]$. We abbreviate $C[t_0, T]$, $C_\sigma[t_0, T]$, $C^1_\sigma[t_0, T]$, $C^{\alpha,\beta}_{1-\gamma}[t_0, T]$ with $\mathcal{C}$, C_σ, C^1_σ, $C^{\alpha,\beta}_{1-\gamma}$, respectively.

Definition 2.1. ([20]) *Let $\alpha \in (0, 1)$. The left-sided Riemann-Liouville fractional integral $I^{\alpha}_{t_0^+}\Psi$ and derivative ${}^{RL}D^{\alpha}_{t_0^+}\Psi$ are defined by*

$$(I^{\alpha}_{t_0^+}\Psi)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha-1}\Psi(s)ds, \quad t > t_0,$$

$$({}^{RL}D^{\alpha}_{t_0^+}\Psi)(t) = \frac{d}{dt}(I^{1-\alpha}_{t_0^+}\Psi)(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_{t_0}^{t} (t - s)^{-\alpha}\Psi(s)ds, \quad t > t_0,$$

provided the right-hand sides exist, where $\Gamma(\cdot)$ denotes the Gamma function.

In [32], the author introduced a modified version of the Hilfer fractional derivative, as described below.

Definition 2.2. ([32]) *The left-sided Hilfer fractional derivative of order* $\alpha \in (0,1), \beta \in [0,1]$ *of* $\Psi(t)$ *is defined by:*

$$^{H}D_{t_0^+}^{\alpha,\beta}\Psi(t) = {}^{RL}D_{t_0^+}^{1-\gamma+\alpha}[(I_{t_0^+}^{1-\gamma}\Psi)(t) - (I_{t_0^+}^{1-\gamma}\Psi)(t_0^+)], \quad t > t_0,$$

where $\gamma = \alpha + \beta(1-\alpha)$.

Lemma 2.1. *Let* $\theta > \gamma$, *then*

$$^{H}D_{t_0^+}^{\alpha,\beta}((t-t_0)^{\theta-1}) = \frac{\Gamma(\theta)}{\Gamma(\theta-\alpha)}(t-t_0)^{\theta-\alpha-1}, \quad t > t_0.$$

In particular, $^{H}D_{t_0^+}^{\alpha,\beta}((t-t_0)^{\gamma-1}) = 0$.

Proof. From [20, Property 2.1], we deduce that

$$\begin{aligned} ^{H}D_{t_0^+}^{\alpha,\beta}((t-t_0)^{\theta-1}) &= {}^{RL}D_{t_0^+}^{1-\gamma+\alpha}[(I_{t_0^+}^{1-\gamma}((s-t_0)^{\theta-1}))(t) - (I_{t_0^+}^{1-\gamma}((s-t_0)^{\theta-1}))(t_0^+)] \\ &= {}^{RL}D_{t_0^+}^{1-\gamma+\alpha}\Big[\frac{\Gamma(\theta)}{\Gamma(\theta-\gamma+1)}(t-t_0)^{\theta-\gamma}\Big] \\ &= \frac{\Gamma(\theta)}{\Gamma(\theta-\alpha)}(t-t_0)^{\theta-\alpha-1}. \end{aligned}$$

In particular, $^{H}D_{t_0^+}^{\alpha,\beta}((t-t_0)^{\gamma-1}) = {}^{RL}D_{t_0^+}^{1-\gamma+\alpha}[\Gamma(\gamma) - \Gamma(\gamma)] = 0.$ □

Lemma 2.2. *Let* $\sigma \in (0,1)$ *and* $\omega > \sigma$, *if* $\Psi \in C_\sigma$, *then* $\big(I_{t_0^+}^{\omega}\Psi\big)(t_0^+) = \lim_{t\to t_0^+}\big(I_{t_0^+}^{\omega}\Psi\big)(t) = 0$.

Proof. The conclusion is an immediate consequence of Definition 2.1 and the following inequality holds:

$$\begin{aligned} |I_{t_0^+}^{\omega}\Psi(t)| &\le \frac{1}{\Gamma(\omega)}\int_{t_0}^{t}(t-s)^{\omega-1}|\Psi(s)|ds \le \frac{1}{\Gamma(\omega)}\int_{t_0}^{t}(t-s)^{\omega-1}(s-t_0)^{-\sigma}ds \cdot \|\Psi\|_{C_\sigma} \\ &= \frac{(t-t_0)^{\omega-\sigma}\Gamma(1-\sigma)}{\Gamma(\omega+1-\sigma)}\|\Psi\|_{C_\sigma}. \end{aligned}$$

□

Lemma 2.3. *If* $\Psi \in C_{1-\gamma}$, *then*

$$({}^{H}D_{t_0^+}^{\alpha,\beta}I_{t_0^+}^{\alpha}\Psi)(t) = \Psi(t).$$

Proof. For $\Psi \in C_{1-\gamma}$, from Lemma 2.2, we deduce $(I_{t_0^+}^{1-\gamma+\alpha}\Psi)(t_0^+) = 0$. From Definition 2.2 and [20, Lemma 2.9(b)], we find that

$$({}^{H}D_{t_0^+}^{\alpha,\beta}I_{t_0^+}^{\alpha}\Psi)(t) = {}^{RL}D_{t_0^+}^{1-\gamma+\alpha}[(I_{t_0^+}^{1-\gamma+\alpha}\Psi)(t) - (I_{t_0^+}^{1-\gamma+\alpha}\Psi)(t_0^+)] = \Psi(t).$$

□

Theorem 2.1. *If* $\Psi \in C_{1-\gamma}^{\alpha,\beta}$ $(\beta \neq 0)$, *then*

$$\big(I_{t_0^+}^{\alpha}\,{}^{H}D_{t_0^+}^{\alpha,\beta}\Psi\big)(t) = \Psi(t) - \frac{(I_{t_0^+}^{1-\gamma}\Psi)(t_0^+)}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}.$$

Proof. Since $\Psi \in C^{\alpha,\beta}_{1-\gamma}$, we find that $\Psi \in C_{1-\gamma}$ and ${}^{H}D^{\alpha,\beta}_{t_0^+}\Psi(t) \in C_{1-\gamma}$, hence

$$I^{\gamma-\alpha}_{t_0^+}[(I^{1-\gamma}_{t_0^+}\Psi)(t) - (I^{1-\gamma}_{t_0^+}\Psi)(t_0^+)] \in C^1_{1-\gamma},$$

and

$$\{I^{\gamma-\alpha}_{t_0^+}[(I^{1-\gamma}_{t_0^+}\Psi)(t) - (I^{1-\gamma}_{t_0^+}\Psi)(t_0^+)]\}(t_0^+) = 0.$$

From [20, Lemma 2.9], it follows that

$$I^{1-\gamma+\alpha}_{t_0^+}\,{}^{RL}D^{1-\gamma+\alpha}_{t_0^+}[(I^{1-\gamma}_{t_0^+}\Psi)(t) - (I^{1-\gamma}_{t_0^+}\Psi)(t_0^+)] = (I^{1-\gamma}_{t_0^+}\Psi)(t) - (I^{1-\gamma}_{t_0^+}\Psi)(t_0^+),$$

thus

$$\begin{aligned}
\big(I^{\alpha}_{t_0^+}\,{}^{H}D^{\alpha,\beta}_{t_0^+}\Psi\big)(t) &= I^{\alpha}_{t_0^+}\,{}^{RL}D^{1-\gamma+\alpha}_{t_0^+}[(I^{1-\gamma}_{t_0^+}\Psi)(t) - (I^{1-\gamma}_{t_0^+}\Psi)(t_0^+)] \\
&= {}^{RL}D^{1-\gamma}_{t_0^+}I^{1-\gamma+\alpha}_{t_0^+}\,{}^{RL}D^{1-\gamma+\alpha}_{t_0^+}[(I^{1-\gamma}_{t_0^+}\Psi)(t) - (I^{1-\gamma}_{t_0^+}\Psi)(t_0^+)] \\
&= {}^{RL}D^{1-\gamma}_{t_0^+}[(I^{1-\gamma}_{t_0^+}\Psi)(t) - (I^{1-\gamma}_{t_0^+}\Psi)(t_0^+)] \\
&= \Psi(t) - \frac{(I^{1-\gamma}_{t_0^+}\Psi)(t_0^+)}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}.
\end{aligned}$$

□

Definition 2.3. ([15, 27]) *For $\mu, \nu > 0$, $y \in \mathbb{R}$, the classical Mittag-Leffler function $E_\mu(y)$ and the generalized Mittag-Leffler function $E_{\mu,\nu}(y)$ are defined by*

$$E_\mu(y) = \sum_{k=0}^{\infty}\frac{y^k}{\Gamma(\mu k+1)}, \quad E_{\mu,\nu}(y) = \sum_{k=0}^{\infty}\frac{y^k}{\Gamma(\mu k+\nu)}.$$

Lemma 2.4. ([15]) *Let $\mu, \nu_1, \nu_2 > 0$, $\omega_1, \omega_2 \in \mathbb{R}$, $\omega_1 \neq \omega_2$, then*

$$\begin{aligned}
&\int_0^t s^{\nu_1-1}E_{\mu,\nu_1}(\omega_1 s^\mu)(t-s)^{\nu_2-1}E_{\mu,\nu_2}(\omega_2(t-s)^\mu)\,ds \\
=\;& \frac{t^{\nu_1+\nu_2-1}}{\omega_1-\omega_2}\Big[\omega_1 E_{\mu,\nu_1+\nu_2}(\omega_1 t^\mu) - \omega_2 E_{\mu,\nu_1+\nu_2}(\omega_2 t^\mu)\Big].
\end{aligned}$$

The three-parameter Mittag-Leffler function (see [27]) is defined by

$$E^{\varsigma}_{\mu,\nu}(y) = \sum_{k=0}^{\infty}\frac{(\varsigma)_k}{\Gamma(\mu k+\nu)}\frac{y^k}{k!},$$

where $(\varsigma)_k = \varsigma(\varsigma+1)\cdots(\varsigma+k-1)$ for $k \geq 1$ and $(\varsigma)_0 = 1$. Furthermore, the corresponding integral operator (see [19]) is defined by

$$(\mathbf{E}^{\varsigma}_{\mu,\nu;\omega}\varphi)(t) = \int_0^t (t-s)^{\nu-1}E^{\varsigma}_{\mu,\nu}[-\omega(t-s)^\mu]\varphi(s)ds.$$

Lemma 2.5. ([41]) *Let φ, $\psi : [0,T) \to [0,\infty)$ be locally integrable and $\kappa(t) : [0,T) \to [0,\infty)$ be a bounded, nondecreasing, and continuous function. If $\kappa(t)$ satisfies*

$$\varphi(t) \leq \psi(t) + \kappa(t)\int_0^t (t-s)^{\theta-1}\varphi(t)ds,$$

then

$$\varphi(t) \le \psi(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(\kappa(t)\Gamma(\theta))^n}{\Gamma(n\theta)}(t-s)^{n\theta-1}\psi(s)ds.$$

In particular, if $\psi(t)$ *is nondecreasing, then* $\varphi(t) \le \psi(t)E_\theta(\kappa(t)\Gamma(\theta)t^\theta)$.

3. Existence and uniqueness results

In this part, we focus on investigating the existence and uniqueness of solutions of the system (2). We begin our investigation by examining the initial value problem (IVP) for a linear equation.

3.1. The linear problem.

Theorem 3.1. *Let* $f(t) \in C_{1-\gamma_1}$. *Then,* $p(t) \in C_{1-\gamma_1}^{\alpha_1,\beta_1}$ *satisfies the following IVP*

$$\begin{cases} {}^H D_{0^+}^{\alpha_2,\beta_2}[{}^H D_{0^+}^{\alpha_1,\beta_1} + \lambda_1(t)]p(t) = f(t),\ t \in J, \\ (I_{0^+}^{1-\gamma_1}p)(0^+) = p_0, \end{cases} \tag{3}$$

if and only if $p(t)$ *satisfies the following integral equation*

$$p(t) = -I_{0^+}^{\alpha_1}[\lambda_1(t)p(t)] + I_{0^+}^{\alpha_1+\alpha_2}f(t) + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0. \tag{4}$$

Proof. Let $p(t) \in C_{1-\gamma_1}^{\alpha_1,\beta_1}$ satisfy (3), then ${}^H D_{0^+}^{\alpha_1,\beta_1}p(t) + \lambda_1(t)p(t) \in C_{1-\gamma_1}^{\alpha_2,\beta_2}$. From Lemma 2.2, one has

$$\{I_{0^+}^{1-\gamma_2}[{}^H D_{0^+}^{\alpha_1,\beta_1}p(t) + \lambda_1(t)p(t)]\}(0^+) = 0.$$

By Theorem 2.1, we find that

$$I_{0^+}^{\alpha_2}\,{}^H D_{0^+}^{\alpha_2,\beta_2}[{}^H D_{0^+}^{\alpha_1,\beta_1}p(t) + \lambda_1(t)p(t)] = {}^H D_{0^+}^{\alpha_1,\beta_1}p(t) + \lambda_1(t)p(t). \tag{5}$$

From (3) and (5), we get

$$[{}^H D_{0^+}^{\alpha_1,\beta_1} + \lambda_1(t)]p(t) = I_{0^+}^{\alpha_2}f(t). \tag{6}$$

Applying $I_{0^+}^{\alpha_1}$ to (6) and using Theorem 2.1, we obtain (4).

If $p(t)$ satisfies (4), clearly, $p(t) \in C_{1-\gamma_1}^{\alpha_1,\beta_1}$ and successive applications of the operators ${}^H D_{0^+}^{\alpha_1,\beta_1}$ and ${}^H D_{0^+}^{\alpha_2,\beta_2}$ to (4) yield the first equation in (3). Moreover,

$$(I_{0^+}^{1-\gamma_1}p)(t) = -I_{0^+}^{1-\gamma_1+\alpha_1}[\lambda_1(t)p(t)] + I_{0^+}^{\alpha_1+\alpha_2+1-\gamma_1}f(t) + p_0.$$

By Lemma 2.2, one obtains $(I_{0^+}^{1-\gamma_1}p)(0^+) = p_0$. Now, we deduce that (3) is equivalent to (4). □

Theorem 3.2. *Let* $f(t) \in C_{1-\gamma_1}$. *Then, the integral equation (4) has a unique solution* $p(t) \in C_{1-\gamma_1}^{\alpha_1,\beta_1}$ *given by:*

$$p(t) = \sum_{k=0}^{\infty}(-1)^k(I_{0^+}^{\alpha_1}\lambda_1(\cdot))^k\big[I_{0^+}^{\alpha_1+\alpha_2}f(t) + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0\big].$$

Proof. We define the following operator $\mathcal{T} : C_{1-\gamma_1} \to C_{1-\gamma_1}$:

$$(\mathcal{T}p)(t) = -I_{0^+}^{\alpha_1}[\lambda_1(t)p(t)] + I_{0^+}^{\alpha_1+\alpha_2}f(t) + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0.$$

It is evident that $\mathcal{T}$ is well-defined, and any fixed point of $\mathcal{T}$ corresponds to a solution of equation (4). Since

$$\int_0^t (t-s)^{\theta_1-1}s^{\theta_2-1}ds = t^{\theta_1+\theta_2-1}\cdot\frac{\Gamma(\theta_1)\Gamma(\theta_2)}{\Gamma(\theta_1+\theta_2)}, \quad 0<\theta_1,\,\theta_2<1, \tag{7}$$

for $p,\,\widetilde{p}\in C_{1-\gamma_1}$, we have

$$\begin{aligned} t^{1-\gamma_1}|(\mathcal{T}p)(t)-(\mathcal{T}\widetilde{p})(t)| &\le \frac{\|\lambda_1\|_{\mathcal{C}}t^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}s^{\gamma_1-1}ds\cdot\|p-\widetilde{p}\|_{C_{1-\gamma_1}} \\ &= \frac{t^{\alpha_1}\|\lambda_1\|_{\mathcal{C}}\Gamma(\gamma_1)}{\Gamma(\alpha_1+\gamma_1)}\cdot\|p-\widetilde{p}\|_{C_{1-\gamma_1}}. \end{aligned}$$

Furthermore, we arrive at

$$\begin{aligned} t^{1-\gamma_1}|(\mathcal{T}^2p)(t)-(\mathcal{T}^2\widetilde{p})(t)| &\le \frac{\|\lambda_1\|_{\mathcal{C}}t^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}|(\mathcal{T}p)(s)-(\mathcal{T}\widetilde{p})(s)|ds \\ &\le \frac{\|\lambda_1\|_{\mathcal{C}}^2\Gamma(\gamma_1)t^{1-\gamma_1}}{\Gamma(\alpha_1+\gamma_1)\Gamma(\alpha_1)}\cdot\int_0^t (t-s)^{\alpha_1-1}s^{\gamma_1+\alpha_1-1}ds\cdot\|p-\widetilde{p}\|_{C_{1-\gamma_1}} \\ &= \frac{\|\lambda_1\|_{\mathcal{C}}^2\Gamma(\gamma_1)t^{2\alpha_1}}{\Gamma(2\alpha_1+\gamma_1)}\cdot\|p-\widetilde{p}\|_{C_{1-\gamma_1}}. \end{aligned}$$

By induction, we derive that

$$t^{1-\gamma_1}|(\mathcal{T}^kp)(t)-(\mathcal{T}^k\widetilde{p})(t)| \le \frac{\|\lambda_1\|_{\mathcal{C}}^k\Gamma(\gamma_1)t^{k\alpha_1}}{\Gamma(k\alpha_1+\gamma_1)}\cdot\|p-\widetilde{p}\|_{C_{1-\gamma_1}}.$$

Then for sufficiently large k, one gets

$$t^{1-\gamma_1}|(\mathcal{T}^kp)(t)-(\mathcal{T}^k\widetilde{p})(t)| < \vartheta\|p-\widetilde{p}\|_{C_{1-\gamma_1}}, \quad \vartheta\in(0,1).$$

Through the application of the generalized Banach contraction principle, it can be conclusively established that the operator $\mathcal{T}$ possesses a single fixed point $\hat{p}\in C_{1-\gamma_1}$, which satisfies (4). Thus the following sequence $\{\hat{p}_n\}$ is convergent in $C_{1-\gamma_1}$:

$$\begin{cases} \hat{p}_0(t) = \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0 + I_{0^+}^{\alpha_1+\alpha_2}f(t), \\ \hat{p}_n(t) = \hat{p}_0(t) - I_{0^+}^{\alpha_1}[\lambda_1(t)\hat{p}_{n-1}(t)], \quad n=1,2,\cdots. \end{cases}$$

Furthermore, we find that

$$\begin{aligned} \hat{p}_1(t) &= \hat{p}_0(t) - I_{0^+}^{\alpha_1}[\lambda_1(t)\hat{p}_0(t)], \\ \hat{p}_2(t) &= \hat{p}_0(t) + \sum_{k=1}^{2}(-1)^k(I_{0^+}^{\alpha_1}\lambda_1(\cdot))^k\hat{p}_0(t), \\ &\cdots \\ \hat{p}_n(t) &= \hat{p}_0(t) + \sum_{k=1}^{n}(-1)^k(I_{0^+}^{\alpha_1}\lambda_1(\cdot))^k\hat{p}_0(t). \end{aligned}$$

Hence, the limit

$$\hat{p}(t) = \lim_{n\to\infty}\sum_{k=0}^{n}(-1)^k(I_{0^+}^{\alpha_1}\lambda_1(\cdot))^k\hat{p}_0(t) = \sum_{k=0}^{\infty}(-1)^k(I_{0^+}^{\alpha_1}\lambda_1(\cdot))^k\hat{p}_0(t),$$

is the unique solution of (4). One can observe that from Theorem 3.1 it follows that the solution belongs to $C^{\alpha_1,\beta_1}_{1-\gamma_1}$. □

The following theorem follows directly from Theorem 3.1 and the proof of Theorem 3.2.

Theorem 3.3. *Let* $\omega(t)\in\mathcal{C}$ *and* $h(t)\in C_{1-\gamma_1}$. *Then, the following IVP*

$$\begin{cases} [{}^HD^{\alpha_1,\beta_1}_{0^+}+\omega(t)]p(t)=h(t),\ t\in J,\\ (I^{1-\gamma_1}_{0^+}p)(0^+)=p_0,\end{cases}$$

has a unique solution $p(t)\in C^{\alpha_1,\beta_1}_{1-\gamma_1}$ *and*

$$p(t)=\sum_{k=0}^{\infty}(-1)^k(I^{\alpha_1}_{0^+}\omega(\cdot))^k\big[I^{\alpha_1}_{0^+}h(t)+\frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0\big].$$

Theorem 3.4. *Let* $h(t)\in C_{1-\gamma_1}$. *Then, the following IVP*

$$\begin{cases} \big[{}^HD^{\alpha_2,\beta_2}_{0^+}+\lambda_2(t)\big]\big[{}^HD^{\alpha_1,\beta_1}_{0^+}+\lambda_1(t)\big]p(t)=h(t),\\ (I^{1-\gamma_1}_{0^+}p)(0^+)=p_0,\end{cases}\tag{8}$$

has a unique solution

$$p(t)=\sum_{k=0}^{\infty}(-1)^k(I^{\alpha_1}_{0^+}\lambda_1(\cdot))^k\Big[I^{\alpha_1}_{0^+}\sum_{n=0}^{\infty}(-1)^n(I^{\alpha_2}_{0^+}\lambda_2(\cdot))^nI^{\alpha_2}_{0^+}h(t)+\frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0\Big].\tag{9}$$

Proof. Let $p(t)\in C^{\alpha_1,\beta_1}_{1-\gamma_1}$ satisfy (8). If we set

$$\zeta(t):=[{}^HD^{\alpha_1,\beta_1}_{0^+}+\lambda_1(t)]p(t),\tag{10}$$

then $\zeta(t)\in C^{\alpha_2,\beta_2}_{1-\gamma_1}$ and

$$\big[{}^HD^{\alpha_2,\beta_2}_{0^+}+\lambda_2(t)\big]\zeta(t)=h(t).\tag{11}$$

From Lemma 2.2, one has

$$(I^{1-\gamma_2}_{0^+}\zeta)(0^+)=0.\tag{12}$$

By Theorem 3.3, system (11),(12) has a unique solution $\zeta(t)$ of the explicit form:

$$\zeta(t)=\sum_{k=0}^{\infty}(-1)^k(I^{\alpha_2}_{0^+}\lambda_2(\cdot))^kI^{\alpha_2}_{0^+}h(t).$$

Through the synthesis of (10) and Theorem 3.3, there exists a unique solution $p(t)$ which is formally given by (9). □

3.2. **The nonlinear system.** In this subsection, we study the nonlinear system (2). Let $X=C^{\alpha_1,\beta_1}_{1-\gamma_1}$, since X is a Banach space, it follows that $X\times X$ is a Banach space with the norm $\|(u,v)\|_{X\times X}=\|u\|_{C^{\alpha_1,\beta_1}_{1-\gamma_1}}+\|v\|_{C^{\alpha_1,\beta_1}_{1-\gamma_1}}$.

We need the following assumption:

(H) The functions $f_j:J\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfy $f_j(t,p(t),{}^HD^{\alpha_1,\beta_1}_{0^+}p(t),q(t),{}^HD^{\alpha_1,\beta_1}_{0^+}q(t))\in C_{1-\gamma_1}(j=1,2)$ for any $(p,q)\in X\times X$ and for $l_i(t),\ m_i(t)\in\mathcal{C},u_i(t),\ v_i(t)\in C_{1-\gamma_1}(i=1,2,3,4)$,

$$\begin{aligned}|f_1(t,u_1(t),u_2(t),u_3(t),u_4(t))-f_1(t,v_1(t),v_2(t),v_3(t),v_4(t))| &\le \sum_{i=1}^{4}l_i(t)|u_i(t)-v_i(t)|,\\ |f_2(t,u_1(t),u_2(t),u_3(t),u_4(t))-f_2(t,v_1(t),v_2(t),v_3(t),v_4(t))| &\le \sum_{i=1}^{4}m_i(t)|u_i(t)-v_i(t)|.\end{aligned}$$

We denote

$$\begin{aligned} M &:= \max\{\|\lambda_1\|_{\mathcal{C}}, \|\delta_1\|_{\mathcal{C}}, \max_{1\le i\le 4}\|l_i\|_{\mathcal{C}}, \max_{1\le i\le 4}\|m_i\|_{\mathcal{C}}\}, && (13)\\ \widehat{M} &:= 5M\Gamma(\gamma_1)\max\Big\{\frac{T^{\alpha_1}}{\Gamma(\alpha_1+\gamma_1)}, \frac{T^{\alpha_2}}{\Gamma(\alpha_2+\gamma_1)}, \frac{T^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+\gamma_1)}\Big\}. && (14)\end{aligned}$$

Theorem 3.5. *Assume that (H) is satisfied, if $\widehat{M} < 1$, the system (2) has a unique solution.*

Proof. From the proof of Theorem 3.1, we see that

$$\begin{cases} p(t) = -I_{0^+}^{\alpha_1}[\lambda_1(t)p(t)] + I_{0^+}^{\alpha_1+\alpha_2}\widetilde{f}_1(t) + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0,\ t\in J,\\ q(t) = -I_{0^+}^{\alpha_1}[\delta_1(t)q(t)] + I_{0^+}^{\alpha_1+\alpha_2}\widetilde{f}_2(t) + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}q_0,\ t\in J,\end{cases} \tag{15}$$

where

$$\begin{aligned}\widetilde{f}_1(t) &= f_1(t, p(t), {}^H D_{0^+}^{\alpha_1,\beta_1}p(t), q(t), {}^H D_{0^+}^{\alpha_1,\beta_1}q(t)),\\ \widetilde{f}_2(t) &= f_2(t, p(t), {}^H D_{0^+}^{\alpha_1,\beta_1}p(t), q(t), {}^H D_{0^+}^{\alpha_1,\beta_1}q(t)).\end{aligned}$$

Next, we prove the system (15) has a unique solution. We define an operator $\mathcal{F}: X\times X \to X\times X$ by

$$\mathcal{F}(p,q)(t) = \Big(\mathcal{F}_1(p,q)(t), \mathcal{F}_2(p,q)(t)\Big),$$

where

$$\begin{aligned}\mathcal{F}_1(p,q)(t) &= -I_{0^+}^{\alpha_1}[\lambda_1(t)p(t)] + I_{0^+}^{\alpha_1+\alpha_2}\widetilde{f}_1(t) + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0,\\ \mathcal{F}_2(p,q)(t) &= -I_{0^+}^{\alpha_1}[\delta_1(t)q(t)] + I_{0^+}^{\alpha_1+\alpha_2}\widetilde{f}_2(t) + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}q_0.\end{aligned}$$

Clearly, $\mathcal{F}$ is well-defined and

$${}^H D_{0^+}^{\alpha_1,\beta_1}\mathcal{F}(p,q)(t) = \Big({}^H D_{0^+}^{\alpha_1,\beta_1}\mathcal{F}_1(p,q)(t), {}^H D_{0^+}^{\alpha_1,\beta_1}\mathcal{F}_2(p,q)(t)\Big),$$

where

$$\begin{aligned}{}^H D_{0^+}^{\alpha_1,\beta_1}\mathcal{F}_1(p,q)(t) &= -\lambda_1(t)p(t) + I_{0^+}^{\alpha_2}\widetilde{f}_1(t),\\ {}^H D_{0^+}^{\alpha_1,\beta_1}\mathcal{F}_2(p,q)(t) &= -\delta_1(t)q(t) + I_{0^+}^{\alpha_2}\widetilde{f}_2(t).\end{aligned}$$

Let $p_j,\ q_j \in X (j = 1, 2)$ satisfy

$$\begin{aligned}(I_{0^+}^{1-\gamma_1}p_1)(0^+) &= (I_{0^+}^{1-\gamma_1}p_2)(0^+) = p_0,\\ (I_{0^+}^{1-\gamma_1}q_1)(0^+) &= (I_{0^+}^{1-\gamma_1}q_2)(0^+) = q_0.\end{aligned}$$

By (7) we deduce

$$\begin{aligned} I_{0^+}^{\alpha_1}|\lambda_1(t)p_1(t) - \lambda_1(t)p_2(t)| &\le \frac{M}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}s^{\gamma_1-1}ds\cdot\|p_1-p_2\|_{C_{1-\gamma_1}}\\ &\le \frac{\widehat{M}}{5}t^{\gamma_1-1}\|p_1-p_2\|_{C_{1-\gamma_1}}, && (16)\end{aligned}$$

and

$$\begin{aligned}|\lambda_1(t)p_1(t)-\lambda_1(t)p_2(t)| &= |\lambda_1(t)[I_{0^+}^{\alpha_1}\,{}^HD_{0^+}^{\alpha_1,\beta_1}p_1(t)-I_{0^+}^{\alpha_1}\,{}^HD_{0^+}^{\alpha_1,\beta_1}p_2(t)]|\\ &\le \frac{M}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}s^{\gamma_1-1}ds\cdot\|{}^HD_{0^+}^{\alpha_1,\beta_1}p_1-{}^HD_{0^+}^{\alpha_1,\beta_1}p_2\|_{C_{1-\gamma_1}}\\ &\le \frac{\widehat{M}}{5}t^{\gamma_1-1}\cdot\|{}^HD_{0^+}^{\alpha_1,\beta_1}p_1-{}^HD_{0^+}^{\alpha_1,\beta_1}p_2\|_{C_{1-\gamma_1}}. \end{aligned}\tag{17}$$

From (H), (13) and (14), we obtain

$$\begin{aligned}&\big|f_1(t,p_1(t),{}^HD_{0^+}^{\alpha_1,\beta_1}p_1(t),q_1(t),{}^HD_{0^+}^{\alpha_1,\beta_1}q_1(t))-f_1(t,p_2(t),{}^HD_{0^+}^{\alpha_1,\beta_1}p_2(t),q_2(t),{}^HD_{0^+}^{\alpha_1,\beta_1}q_2(t))\big|\\ &\le l_1(t)|p_1(t)-p_2(t)|+l_2(t)|{}^HD_{0^+}^{\alpha_1,\beta_1}p_1(t)-{}^HD_{0^+}^{\alpha_1,\beta_1}p_2(t)|+l_3(t)|q_1(t)-q_2(t)|\\ &+l_4(t)|{}^HD_{0^+}^{\alpha_1,\beta_1}q_1(t)-{}^HD_{0^+}^{\alpha_1,\beta_1}q_2(t)|\\ &\le Mt^{\gamma_1-1}\|(p_1-p_2,q_1-q_2)\|_{X\times X},\end{aligned}$$

and similarly

$$\begin{aligned}&|f_2(t,p_1(t),{}^HD_{0^+}^{\alpha_1,\beta_1}p_1(t),q_1(t),{}^HD_{0^+}^{\alpha_1,\beta_1}q_1(t))-f_2(t,p_2(t),{}^HD_{0^+}^{\alpha_1,\beta_1}p_2(t),q_2(t),{}^HD_{0^+}^{\alpha_1,\beta_1}q_2(t))|\\ &\le Mt^{\gamma_1-1}\|(p_1-p_2,q_1-q_2)\|_{X\times X}.\end{aligned}$$

Hence for $j=1,2$

$$\begin{aligned}&|I_{0^+}^{\alpha_2}[f_j(t,p_1(t),{}^HD_{0^+}^{\alpha_1,\beta_1}p_1(t),q_1(t),{}^HD_{0^+}^{\alpha_1,\beta_1}q_1(t))-f_j(t,p_2(t),{}^HD_{0^+}^{\alpha_1,\beta_1}p_2(t),q_2(t),{}^HD_{0^+}^{\alpha_1,\beta_1}q_2(t))]|\\ &\le \frac{M}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}s^{\gamma_1-1}ds\cdot\|(p_1-p_2,q_1-q_2)\|_{X\times X}\\ &\le \frac{\widehat{M}}{5}t^{\gamma_1-1}\cdot\|(p_1-p_2,q_1-q_2)\|_{X\times X},\end{aligned}\tag{18}$$

and

$$\begin{aligned}&|I_{0^+}^{\alpha_1+\alpha_2}[f_j(t,p_1(t),{}^HD_{0^+}^{\alpha_1,\beta_1}p_1(t),q_1(t),{}^HD_{0^+}^{\alpha_1,\beta_1}q_1(t))-f_j(t,p_2(t),{}^HD_{0^+}^{\alpha_1,\beta_1}p_2(t),q_2(t),{}^HD_{0^+}^{\alpha_1,\beta_1}q_2(t))]|\\ &\le \frac{M}{\Gamma(\alpha_1+\alpha_2)}\int_0^t (t-s)^{\alpha_1+\alpha_2-1}s^{\gamma_1-1}ds\cdot\|(p_1-p_2,q_1-q_2)\|_{X\times X}\\ &\le \frac{\widehat{M}}{5}t^{\gamma_1-1}\cdot\|(p_1-p_2,q_1-q_2)\|_{X\times X}.\end{aligned}\tag{19}$$

Similarly, we can deduce

$$I_{0^+}^{\alpha_1}|\delta_1(t)q_1(t)-\delta_1(t)q_2(t)| \le \frac{\widehat{M}}{5}t^{\gamma_1-1}\|q_1-q_2\|_{C_{1-\gamma_1}},\tag{20}$$

$$|\delta_1(t)q_1(t)-\delta_1(t)q_2(t)| \le \frac{\widehat{M}}{5}t^{\gamma_1-1}\cdot\|{}^HD_{0^+}^{\alpha_1,\beta_1}q_1-{}^HD_{0^+}^{\alpha_1,\beta_1}q_2\|_{C_{1-\gamma_1}}.\tag{21}$$

Now from (16)-(21), we find

$$\|\mathcal{F}_1(p_1,q_1)-\mathcal{F}_1(p_2,q_2)\|_{C_{1-\gamma_1}^{\alpha_1,\beta_1}} \le \frac{\widehat{M}}{5}\big[\|p_1-p_2\|_{C_{1-\gamma_1}^{\alpha_1,\beta_1}}+\|(p_1-p_2,q_1-q_2)\|_{X\times X}\big],$$

and

$$\|\mathcal{F}_2(p_1,q_1)-\mathcal{F}_2(p_2,q_2)\|_{C_{1-\gamma_1}^{\alpha_1,\beta_1}} \le \frac{\widehat{M}}{5}\big[\|q_1-q_2\|_{C_{1-\gamma_1}^{\alpha_1,\beta_1}}+\|(p_1-p_2,q_1-q_2)\|_{X\times X}\big].$$

Thus

$$\|(\mathcal{F}_1(p_1,q_1)-\mathcal{F}_1(p_2,q_2),\ \mathcal{F}_2(p_1,q_1)-\mathcal{F}_2(p_2,q_2))\|_{X\times X} \le \widehat{M}\|(p_1-p_2,q_1-q_2)\|_{X\times X}.$$

By virtue of the Banach fixed point theorem, $\mathcal{F}$ possesses a unique fixed point $(p, q) \in X \times X$ satisfying (2). $\square$

4. Stability results

This section analyzes the stability properties of (2), beginning with the definitions of UH and UHML stability.

Definition 4.1. *The system (2) is UH stable if there is a constant $C_U > 0$ such that for any $\varepsilon > 0$ and $\widetilde{U}(t) = (\widetilde{p}(t), \widetilde{q}(t)) \in X \times X$ satisfying*

$$\begin{cases} \left| {}^{H}D_{0^+}^{\alpha_2,\beta_2}[{}^{H}D_{0^+}^{\alpha_1,\beta_1} + \lambda_1(t)]\widetilde{p}(t) - f_1(t, \widetilde{p}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{p}(t), \widetilde{q}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{q}(t)) \right| \le \varepsilon, \ t \in J, \\ \left| {}^{H}D_{0^+}^{\alpha_2,\beta_2}[{}^{H}D_{0^+}^{\alpha_1,\beta_1} + \delta_1(t)]\widetilde{q}(t) - f_2(t, \widetilde{p}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{p}(t), \widetilde{q}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{q}(t)) \right| \le \varepsilon, \ t \in J, \end{cases} \tag{22}$$

there exists a solution $U(t) = (p(t), q(t)) \in X \times X$ to the system (2) satisfying

$$\|\widetilde{U} - U\|_{X\times X} \le C_U \varepsilon.$$

Definition 4.2. *The system (2) is UHML stable with respect to $E_{\alpha_2}(t^{\alpha_2})$ if there is a constant $M_{E_{\alpha_2}} > 0$ such that for any $\varepsilon > 0$ and $\widetilde{U}(t) = (\widetilde{p}(t), \widetilde{q}(t)) \in X \times X$ satisfying*

$$\begin{cases} \left| {}^{H}D_{0^+}^{\alpha_2,\beta_2}[{}^{H}D_{0^+}^{\alpha_1,\beta_1} + \lambda_1(t)]\widetilde{p}(t) - f_1(t, \widetilde{p}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{p}(t), \widetilde{q}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{q}(t)) \right| \le \varepsilon E_{\alpha_2}(t^{\alpha_2}), \ t \in J, \\ \left| {}^{H}D_{0^+}^{\alpha_2,\beta_2}[{}^{H}D_{0^+}^{\alpha_1,\beta_1} + \delta_1(t)]\widetilde{q}(t) - f_2(t, \widetilde{p}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{p}(t), \widetilde{q}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{q}(t)) \right| \le \varepsilon E_{\alpha_2}(t^{\alpha_2}), \ t \in J, \end{cases} \tag{23}$$

there exists a solution $U(t) = (p(t), q(t))$ for the system (2) satisfying

$$\|\widetilde{U}(t) - U(t)\|_1 \le \varepsilon M_{E_{\alpha_2}} E_{\alpha_2}(t^{\alpha_2}), \ t \in J,$$

where $\|\cdot\|_1$ denotes the vector norm $\|(\varpi_1(t), \varpi_2(t))\|_1 = |\varpi_1(t)| + |\varpi_2(t)|$.

Remark 4.1. *$\varrho(t) = (\varrho_1(t), \varrho_2(t))$ is termed a solution of (22) if there exists a function $\omega(t) = (\omega_1(t), \omega_2(t))$ (depending on ϱ) such that*

(i) $|\omega_1(t)| \le \varepsilon, \ |\omega_2(t)| \le \varepsilon, \quad$ for all $\quad t \in J$,
(ii) for $t \in J$

$$\begin{cases} {}^{H}D_{0^+}^{\alpha_2,\beta_2}[{}^{H}D_{0^+}^{\alpha_1,\beta_1} + \lambda_1(t)]\varrho_1(t) = f_1(t, \varrho_1(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\varrho_1(t), \varrho_2(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\varrho_2(t)) + \omega_1(t), \\ {}^{H}D_{0^+}^{\alpha_2,\beta_2}[{}^{H}D_{0^+}^{\alpha_1,\beta_1} + \delta_1(t)]\varrho_2(t) = f_2(t, \varrho_1(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\varrho_1(t), \varrho_2(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\varrho_2(t)) + \omega_2(t). \end{cases} \tag{24}$$

Next, we discuss UHML stability of the system (2).

Remark 4.2. *$\varrho(t) = (\varrho_1(t), \varrho_2(t))$ is called a solution of (23) if there exists a function $\omega(t) = (\omega_1(t), \omega_2(t))$ (depending on ϱ) such that $|\omega_j(t)| \le \varepsilon E_{\alpha_2}(t^{\alpha_2}) (j = 1, 2)$ and (24) holds.*

Theorem 4.1. *Assume that (H) holds, if $\widehat{M} < 1$ in (14), then the system (2) is UHML stable.*

Proof. Let $(I_{0^+}^{1-\gamma_1}\widetilde{p})(0^+) = (I_{0^+}^{1-\gamma_1}p)(0^+)$, $(I_{0^+}^{1-\gamma_1}\widetilde{q})(0^+) = (I_{0^+}^{1-\gamma_1}q)(0^+)$, $\widetilde{U}(t) = (\widetilde{p}(t), \widetilde{q}(t))$ be a solution of (23) and denote

$$\begin{cases} \omega_1(t) = {}^{H}D_{0^+}^{\alpha_2,\beta_2}[{}^{H}D_{0^+}^{\alpha_1,\beta_1} + \lambda_1(t)]\widetilde{p}(t) - f_1(t, \widetilde{p}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{p}(t), \widetilde{q}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{q}(t)), \\ \omega_2(t) = {}^{H}D_{0^+}^{\alpha_2,\beta_2}[{}^{H}D_{0^+}^{\alpha_1,\beta_1} + \delta_1(t)]\widetilde{p}(t) - f_2(t, \widetilde{p}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{p}(t), \widetilde{q}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{q}(t)), \end{cases}$$

then

$$|\omega_1(t)| \le \varepsilon E_{\alpha_2}(t^{\alpha_2}), \ |\omega_2(t)| \le \varepsilon E_{\alpha_2}(t^{\alpha_2}).$$

From Theorem 3.1, we deduce that

$$\begin{cases} \widetilde{p}(t) = -I_{0^+}^{\alpha_1}[\lambda_1(t)\widetilde{p}(t)] + I_{0^+}^{\alpha_1+\alpha_2}[f_1(t, \widetilde{p}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{p}(t), \widetilde{q}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{q}(t)) + \omega_1(t)] + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0, \\ \widetilde{q}(t) = -I_{0^+}^{\alpha_1}[\delta_1(t)\widetilde{q}(t)] + I_{0^+}^{\alpha_1+\alpha_2}[f_2(t, \widetilde{p}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{p}(t), \widetilde{q}(t), {}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{q}(t)) + \omega_2(t)] + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}q_0. \end{cases} \tag{25}$$

When we set

$$\begin{aligned}
\Lambda_1(t) &:= |\widetilde{p}(t)-p(t)|,\\
\Lambda_2(t) &:= |{}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{p}(t)-{}^{H}D_{0^+}^{\alpha_1,\beta_1}p(t)|,\\
\Lambda_3(t) &:= |\widetilde{q}(t)-q(t)|,\\
\Lambda_4(t) &:= |{}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{q}(t)-{}^{H}D_{0^+}^{\alpha_1,\beta_1}q(t)|,\\
\varphi(t) &:= \sum_{i=1}^{4}\Lambda_i(t),
\end{aligned}$$

and consider the fact that

$$I_{0^+}^{\alpha_2}E_{\alpha_2}(t^{\alpha_2})=\frac{1}{\Gamma(\alpha_2)}\int_0^t(t-s)^{\alpha_2-1}\sum_{k=0}^{\infty}\frac{s^{\alpha_2k}}{\Gamma(\alpha_2k+1)}ds=\sum_{k=0}^{\infty}\frac{t^{\alpha_2(k+1)}}{\Gamma(\alpha_2(k+1)+1)}<E_{\alpha_2}(t^{\alpha_2}),\quad(26)$$

$$I_{0^+}^{\alpha_1+\alpha_2}E_{\alpha_2}(t^{\alpha_2})\le\frac{T^{\alpha_1}}{\Gamma(\alpha_1+\alpha_2)}\int_0^t(t-s)^{\alpha_2-1}E_{\alpha_2}(s^{\alpha_2})ds<\frac{T^{\alpha_1}\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}E_{\alpha_2}(t^{\alpha_2}),\quad(27)$$

we find that

$$\begin{aligned}
\Lambda_1(t) \le{}& \left|\widetilde{p}(t)+I_{0^+}^{\alpha_1}[\lambda_1(t)\widetilde{p}(t)]-I_{0^+}^{\alpha_1+\alpha_2}[f_1(t,\widetilde{p}(t),{}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{p}(t),\widetilde{q}(t),{}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{q}(t))]-\frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0\right|\\
&+I_{0^+}^{\alpha_1}[|\lambda_1(t)||p(t)-\widetilde{p}(t)|]\\
&+\Big|\{I_{0^+}^{\alpha_1+\alpha_2}[f_1(\cdot,p(\cdot),{}^{H}D_{0^+}^{\alpha_1,\beta_1}p(\cdot),q(\cdot),{}^{H}D_{0^+}^{\alpha_1,\beta_1}q(\cdot))\\
&-f_1(\cdot,\widetilde{p}(\cdot),{}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{p}(\cdot),\widetilde{q}(\cdot),{}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{q}(\cdot))]\}(t)\Big|\\
\le{}& \varepsilon I_{0^+}^{\alpha_1+\alpha_2}E_{\alpha_2}(t^{\alpha_2})+MI_{0^+}^{\alpha_1}\Lambda_1(t)+\frac{MT^{\alpha_1}}{\Gamma(\alpha_1+\alpha_2)}\int_0^t(t-s)^{\alpha_2-1}\varphi(s)ds,\qquad(28)
\end{aligned}$$

where the first term subsequent to the above inequality originates from the combination of (25) and (27), and the third term is obtained directly from assumption (H).

Similarly, one obtains

$$\begin{aligned}
\Lambda_2(t) \le{}& \left|{}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{p}(t)+\lambda_1(t)\widetilde{p}(t)-I_{0^+}^{\alpha_2}f_1(t,\widetilde{p}(t),{}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{p}(t),\widetilde{q}(t),{}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{q}(t))\right|\\
&+|\lambda_1(t)||\widetilde{p}(t)-p(t)|\\
&+\frac{1}{\Gamma(\alpha_2)}\int_0^t(t-s)^{\alpha_2-1}|f_1(s,p(s),{}^{H}D_{0^+}^{\alpha_1,\beta_1}p(s),q(s),{}^{H}D_{0^+}^{\alpha_1,\beta_1}q(s))\\
&-f_1(s,\widetilde{p}(s),{}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{p}(s),\widetilde{q}(s),{}^{H}D_{0^+}^{\alpha_1,\beta_1}\widetilde{q}(s))|ds\\
\le{}& \varepsilon I_{0^+}^{\alpha_2}E_{\alpha_2}(t^{\alpha_2})+MI_{0^+}^{\alpha_1}\Lambda_2(t)+\frac{M}{\Gamma(\alpha_2)}\int_0^t(t-s)^{\alpha_2-1}\varphi(s)ds,\qquad(29)
\end{aligned}$$

and

$$\Lambda_3(t)\le\varepsilon I_{0^+}^{\alpha_1+\alpha_2}E_{\alpha_2}(t^{\alpha_2})+MI_{0^+}^{\alpha_1}\Lambda_3(t)+\frac{MT^{\alpha_1}}{\Gamma(\alpha_1+\alpha_2)}\int_0^t(t-s)^{\alpha_2-1}\varphi(s)ds,\quad(30)$$

$$\Lambda_4(t) \le \varepsilon I_{0^+}^{\alpha_2} E_{\alpha_2}(t^{\alpha_2}) + M I_{0^+}^{\alpha_1} \Lambda_4(t) + \frac{M}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2 - 1} \varphi(s) ds. \tag{31}$$

In view of (26), (27) and (28)-(31), we arrive at

$$\varphi(t) \le \varepsilon C E_{\alpha_2}(t^{\alpha_2}) + \kappa \int_0^t (t-s)^{\alpha_2-1} \varphi(s) ds,$$

where $C := 4 \max\{\frac{T^{\alpha_1}\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}, 1\}$ and $\kappa := 3M \max\{\frac{2T^{\alpha_1}}{\Gamma(\alpha_1+\alpha_2)}, \frac{2}{\Gamma(\alpha_2)}, \frac{T^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1)}\}$.

By Lemma 2.5, we obtain

$$\|\widetilde{U}(t) - U(t)\|_1 \le \varphi(t) \le \varepsilon C E_{\alpha_2}(t^{\alpha_2}) E_{\alpha_2}(\kappa \Gamma(\alpha_2) t^{\alpha_2}) \le \varepsilon M_{E_{\alpha_2}} E_{\alpha_2}(t^{\alpha_2}),$$

where $M_{E_{\alpha_2}} := C E_{\alpha_2}(\kappa \Gamma(\alpha_2) T^{\alpha_2})$. Thus, the system (2) is UHML stable. □

Theorem 4.2. *Let (H) hold, if $\widehat{M} < 1$, the system (2) is UH stable.*

Proof. The proof is similar to those of Theorem 4.1, we omit the details. □

5. Special cases

In this section, we consider some special cases of the system (2).

5.1. **Case I.** Let

$$\begin{aligned} f_1(t, p(t), {}^H D_{0^+}^{\alpha_1,\beta_1} p(t), q(t), {}^H D_{0^+}^{\alpha_1,\beta_1} q(t)) &:= -\lambda_2(t)\big[{}^H D_{0^+}^{\alpha_1,\beta_1} + \lambda_1(t)\big]p(t) + g_1(t, q(t), {}^H D_{0^+}^{\alpha_1,\beta_1} q(t)), \\ f_2(t, p(t), {}^H D_{0^+}^{\alpha_1,\beta_1} p(t), q(t), {}^H D_{0^+}^{\alpha_1,\beta_1} q(t)) &:= -\delta_2(t)\big[{}^H D_{0^+}^{\alpha_1,\beta_1} + \delta_1(t)\big]q(t) + g_2(t, p(t), {}^H D_{0^+}^{\alpha_1,\beta_1} p(t)). \end{aligned}$$

We need the following assumption:

(H') Let $g_j(t, \eta(t), {}^H D_{0^+}^{\alpha_1,\beta_1} \eta(t)) \in C_{1-\gamma_1} (j = 1, 2)$ for any $\eta \in C_{1-\gamma_1}^{\alpha_1,\beta_1}$ and there exist $\widetilde{l}_i(t), \widetilde{m}_i(t) \in \mathcal{C} (i = 1, 2)$ such that

$$|g_1(t, u_1(t), u_2(t)) - g_1(t, \overline{u}_1(t), \overline{u}_2(t))| \le \sum_{i=1}^{2} \widetilde{l}_i(t) |u_i(t) - \overline{u}_i(t)|, \quad u_i(t), \overline{u}_i(t) \in C_{1-\gamma_1} (i = 1, 2),$$

$$|g_2(t, v_1(t), v_2(t)) - g_2(t, \overline{v}_1(t), \overline{v}_2(t))| \le \sum_{i=1}^{2} \widetilde{m}_i(t) |v_i(t) - \overline{v}_i(t)|, \quad v_i(t), \overline{v}_i(t) \in C_{1-\gamma_1} (i = 1, 2).$$

We denote

$$\begin{aligned} M^* &:= \max\{\max_{1\le i\le 2} \|\lambda_i\|_{\mathcal{C}}, \max_{1\le i\le 2} \|\delta_i\|_{\mathcal{C}}, \|\lambda_1\lambda_2\|_{\mathcal{C}}, \max_{1\le i\le 2} \|\widetilde{l}_i\|_{\mathcal{C}}, \max_{1\le i\le 2} \|\widetilde{m}_i\|_{\mathcal{C}}\}, \\ \widetilde{M} &:= 5M^*\Gamma(\gamma_1) \max\{\frac{T^{\alpha_1}}{\Gamma(\alpha_1+\gamma_1)}, \frac{T^{\alpha_2}}{\Gamma(\alpha_2+\gamma_1)}, \frac{T^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+\gamma_1)}\}. \end{aligned}$$

Theorem 5.1. *Assume that (H') holds and $\widetilde{M} < 1$, then the system*

$$\begin{cases} \big[{}^H D_{0^+}^{\alpha_2,\beta_2} + \lambda_2(t)\big]\big[{}^H D_{0^+}^{\alpha_1,\beta_1} + \lambda_1(t)\big] p(t) = g_1(t, q(t), {}^H D_{0^+}^{\alpha_1,\beta_1} q(t)), \\ \big[{}^H D_{0^+}^{\alpha_2,\beta_2} + \delta_2(t)\big]\big[{}^H D_{0^+}^{\alpha_1,\beta_1} + \delta_1(t)\big] q(t) = g_2(t, p(t), {}^H D_{0^+}^{\alpha_1,\beta_1} p(t)), \\ (I_{0^+}^{1-\gamma_1} p)(0^+) = p_0, \quad (I_{0^+}^{1-\gamma_1} q)(0^+) = q_0, \end{cases}$$

has a unique solution and is UH stable and UHML stable with respect to $E_{\alpha_2}(t^{\alpha_2})$.

Proof. Based on Theorem 3.5, the above mentioned system has a unique solution $(p, q) \in X \times X$. Analogously to the proof of Theorem 4.1, we can infer the conclusion. □

In what follows, we consider the case in which $\lambda_i(t) \equiv \lambda_i$ and $\delta_i(t) \equiv \delta_i$ for $i = 1, 2$. For our purposes, we first restate the lemma below.

Lemma 5.1. ([33]) *For $\lambda \in \mathbb{R}$ and $\theta \in (0, 1)$, the operator $I + \lambda I^{\theta}_{t_0^+} : C_{1-\gamma} \to C_{1-\gamma}$ is invertible and bounded, and*

$$(I + \lambda I^{\theta}_{t_0^+})^{-1} h(t) = \sum_{k=0}^{\infty} (-\lambda)^k (I^{k\theta}_{t_0^+}) h(t).$$

From Lemma 5.1, we deduce

$$\begin{aligned}
(I + \lambda I^{\theta}_{t_0^+})^{-1} I^{\theta}_{t_0^+} h(t) &= \sum_{k=0}^{\infty} (-\lambda)^k \int_{t_0}^{t} \frac{(t-s)^{\theta(k+1)-1}}{\Gamma(\theta(k+1))} h(s) ds \\
&= \int_{t_0}^{t} \sum_{k=0}^{\infty} \frac{(-\lambda)^k (t-s)^{\theta(k+1)-1}}{\Gamma(\theta(k+1))} h(s) ds \\
&= \int_{t_0}^{t} (t-s)^{\theta-1} E_{\theta,\theta}(-\lambda(t-s)^{\theta}) h(s) ds. \qquad (32)
\end{aligned}$$

Corollary 5.1. *Let $g_j(t, \cdot, \cdot) : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} (j = 1, 2)$ satisfy (H'), if $\widetilde{M} < 1$, then the Langevin system*

$$\begin{cases}
\left[{}^H D^{\alpha_2,\beta_2}_{0^+} + \lambda_2\right] \left[{}^H D^{\alpha_1,\beta_1}_{0^+} + \lambda_1\right] p(t) = g_1(t, q(t), {}^H D^{\alpha_1,\beta_1}_{0^+} q(t)), \\
\left[{}^H D^{\alpha_2,\beta_2}_{0^+} + \delta_2\right] \left[{}^H D^{\alpha_1,\beta_1}_{0^+} + \delta_1\right] q(t) = g_2(t, p(t), {}^H D^{\alpha_1,\beta_1}_{0^+} p(t)), \\
(I^{1-\gamma_1}_{0^+} p)(0^+) = p_0, \quad (I^{1-\gamma_1}_{0^+} q)(0^+) = q_0,
\end{cases} \qquad (33)$$

has a unique solution

$$\begin{cases}
p(t) = t^{\gamma_1 - 1} E_{\alpha_1,\gamma_1}(-\lambda_1 t^{\alpha_1}) p_0 + \int_0^t (t-s)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(-\lambda_1 (t-s)^{\alpha_1}) \\
\times \int_0^s (s-\tau)^{\alpha_2 - 1} E_{\alpha_2,\alpha_2}(-\lambda_2 (s-\tau)^{\alpha_2}) g_1(\tau, q(\tau), {}^H D^{\alpha_1,\beta_1}_{0^+} q(\tau)) d\tau ds, \\
q(t) = t^{\gamma_1 - 1} E_{\alpha_1,\gamma_1}(-\delta_1 t^{\alpha_1}) q_0 + \int_0^t (t-s)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(-\delta_1 (t-s)^{\alpha_1}) \\
\times \int_0^s (s-\tau)^{\alpha_2 - 1} E_{\alpha_2,\alpha_2}(-\delta_2 (s-\tau)^{\alpha_2}) g_2(\tau, p(\tau), {}^H D^{\alpha_1,\beta_1}_{0^+} p(\tau)) d\tau ds,
\end{cases} \qquad (34)$$

and system (33) is UH stable and UHML stable with respect to $E_{\alpha_2}(t^{\alpha_2})$.

Proof. From Theorem 3.5, it follows that the unique solution to (33) is represented by:

$$\begin{cases}
p(t) = -\lambda_1 I^{\alpha_1}_{0^+} p(t) + I^{\alpha_1+\alpha_2}_{0^+} \widetilde{g}_1(t) + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)} p_0, \ t \in J, \\
q(t) = -\delta_1 I^{\alpha_1}_{0^+} q(t) + I^{\alpha_1+\alpha_2}_{0^+} \widetilde{g}_2(t) + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)} q_0, \ t \in J,
\end{cases} \qquad (35)$$

where

$$\begin{aligned}
\widetilde{g}_1(t) &= -\lambda_2 \left[{}^H D^{\alpha_1,\beta_1}_{0^+} + \lambda_1\right] p(t) + g_1(t, q(t), {}^H D^{\alpha_1,\beta_1}_{0^+} q(t)), \\
\widetilde{g}_2(t) &= -\delta_2 \left[{}^H D^{\alpha_1,\beta_1}_{0^+} + \delta_1\right] q(t) + g_2(t, p(t), {}^H D^{\alpha_1,\beta_1}_{0^+} p(t)),
\end{aligned}$$

and $(p(t), q(t)) \in X \times X$.

Set $\xi(t) := ({}^H D^{\alpha_1,\beta_1}_{0^+} + \lambda_1) p(t) \in C_{1-\gamma_1}$, by (35) we obtain

$$\xi(t) = -\lambda_2 I^{\alpha_2}_{0^+} \xi(t) + I^{\alpha_2}_{0^+} g_1(t, q(t), {}^H D^{\alpha_1,\beta_1}_{0^+} q(t))$$

and $(I_{0^+}^{1-\gamma_2}\xi)(0^+)=0$. By Lemma 5.1 and (32), we get

$$\begin{aligned}\xi(t) &= \sum_{k=0}^{\infty}(-\lambda_2)^k(I_{0^+}^{k\alpha_2})I_{0^+}^{\alpha_2}g_1(t,q(t),\,{}^HD_{0^+}^{\alpha_1,\beta_1}q(t))\\ &= \int_0^t(t-s)^{\alpha_2-1}E_{\alpha_2,\alpha_2}(-\lambda_2(t-s)^{\alpha_2})g_1(s,q(s),\,{}^HD_{0^+}^{\alpha_1,\beta_1}q(s))ds. \qquad (36)\end{aligned}$$

Moreover, by Lemma 5.1 and (32), we have

$$\begin{aligned}(I+\lambda_1I_{0^+}^{\alpha_1})^{-1}\frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0 &= \sum_{k=0}^{\infty}\frac{(-\lambda_1)^k}{\Gamma(\alpha_1k)\Gamma(\gamma_1)}\int_0^t(t-s)^{k\alpha_1-1}s^{\gamma_1-1}ds\cdot p_0\\ &= t^{\gamma_1-1}E_{\alpha_1,\gamma_1}(-\lambda_1t^{\alpha_1})p_0, \qquad (37)\end{aligned}$$

and

$$(I+\lambda_1I_{0^+}^{\alpha_1})^{-1}I_{0^+}^{\alpha_1}\xi(t)=\int_0^t(t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}[-\lambda_1(t-s)^{\alpha_1}]\xi(s)ds. \qquad (38)$$

From the first equation in (35), (37) and (38), we obtain

$$\begin{aligned}p(t) &= \Big(I+\lambda_1I_{0^+}^{\alpha_1}\Big)^{-1}\Big\{\frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0+I_{0^+}^{\alpha_1+\alpha_2}[-\lambda_2\xi(t)+g_1(t,q(t),\,{}^HD_{0^+}^{\alpha_1,\beta_1}q(t))]\Big\}\\ &= \Big(I+\lambda_1I_{0^+}^{\alpha_1}\Big)^{-1}\Big[\frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0+I_{0^+}^{\alpha_1}\xi(t)\Big]\\ &= t^{\gamma_1-1}E_{\alpha_1,\gamma_1}(-\lambda_1t^{\alpha_1})p_0+\int_0^t(t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}[-\lambda_1(t-s)^{\alpha_1}]\xi(s)ds.\end{aligned}$$

Taking (36) into account, we derive the first expression in (34). In a similar way, we obtain the second expression in (34). Theorems 4.1 and 4.2 collectively establish that the system (33) exhibits UH stability and UHML stability with respect to $E_{\alpha_2}(t^{\alpha_2})$. □

5.2. **Case II.**

Let

$$f_1(t,p(t),\,{}^HD_{0^+}^{\alpha_1,\beta_1}p(t),q(t),\,{}^HD_{0^+}^{\alpha_1,\beta_1}q(t)):=-\delta_2(t)\big[{}^HD_{0^+}^{\alpha_1,\beta_1}+\lambda_1(t)\big]p(t)-\lambda_2(t)\big[{}^HD_{0^+}^{\alpha_1,\beta_1}+\delta_1(t)\big]q(t)+h_1(t),$$

$$f_2(t,p(t),\,{}^HD_{0^+}^{\alpha_1,\beta_1}p(t),q(t),\,{}^HD_{0^+}^{\alpha_1,\beta_1}q(t)):=-\lambda_2(t)\big[{}^HD_{0^+}^{\alpha_1,\beta_1}+\delta_1(t)\big]q(t)-\delta_2(t)\big[{}^HD_{0^+}^{\alpha_1,\beta_1}+\lambda_1(t)\big]p(t)+h_2(t),$$

and

$$\begin{aligned}M^{\diamond} &:= \max\{\|\lambda_2\|_{\mathcal{C}},\,\|\delta_2\|_{\mathcal{C}},\,\|\lambda_2\delta_1\|_{\mathcal{C}},\,\|\lambda_1\delta_2\|_{\mathcal{C}}\},\\ \overline{M} &:= 5M^{\diamond}\Gamma(\gamma_1)\max\{\frac{T^{\alpha_1}}{\Gamma(\alpha_1+\gamma_1)},\,\frac{T^{\alpha_2}}{\Gamma(\alpha_2+\gamma_1)},\,\frac{T^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+\gamma_1)}\}.\end{aligned}$$

Theorem 5.2. *Let $h_j(t)\in C_{1-\gamma_1}(j=1,2)$ and $\overline{M}<1$, then the system*

$$\begin{cases}\big[{}^HD_{0^+}^{\alpha_2,\beta_2}+\delta_2(t)\big]\big[{}^HD_{0^+}^{\alpha_1,\beta_1}+\lambda_1(t)\big]p(t)+\lambda_2(t)\big[{}^HD_{0^+}^{\alpha_1,\beta_1}+\delta_1(t)\big]q(t)=h_1(t),\ t\in J,\\ \big[{}^HD_{0^+}^{\alpha_2,\beta_2}+\lambda_2(t)\big]\big[{}^HD_{0^+}^{\alpha_1,\beta_1}+\delta_1(t)\big]q(t)+\delta_2(t)\big[{}^HD_{0^+}^{\alpha_1,\beta_1}+\lambda_1(t)\big]p(t)=h_2(t),\ t\in J,\\ (I_{0^+}^{1-\gamma_1}p)(0^+)=p_0,\quad (I_{0^+}^{1-\gamma_1}q)(0^+)=q_0,\end{cases} \qquad (39)$$

has a unique solution

$$\begin{cases} p(t) = \sum\limits_{k=0}^{\infty}(-I_{0^+}^{\alpha_1}\lambda_1(\cdot))^k\Big\{I_{0^+}^{\alpha_1}\sum\limits_{n=0}^{\infty}(-1)^n\big[I_{0^+}^{\alpha_2}(\lambda_2(\cdot)+\delta_2(\cdot))\big]^n I_{0^+}^{\alpha_2}\big[\lambda_2(t)I_{0^+}^{\alpha_2}(h_1(t)-h_2(t))+h_1(t)\big]\Big\} \\ \quad + \sum\limits_{k=0}^{\infty}(-I_{0^+}^{\alpha_1}\lambda_1(\cdot))^k\frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0, \\ q(t) = \sum\limits_{k=0}^{\infty}(-I_{0^+}^{\alpha_1}\delta_1(\cdot))^k\Big\{I_{0^+}^{\alpha_1}\sum\limits_{n=0}^{\infty}(-1)^n\big[I_{0^+}^{\alpha_2}(\lambda_2(\cdot)+\delta_2(\cdot))\big]^n I_{0^+}^{\alpha_2}\big[-\delta_2(t)I_{0^+}^{\alpha_2}(h_1(t)-h_2(t))+h_2(t)\big]\Big\} \\ \quad + \sum\limits_{k=0}^{\infty}(-I_{0^+}^{\alpha_1}\delta_1(\cdot))^k\frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}q_0, \end{cases} \tag{40}$$

and the system (39) is UH stable and UHML stable with respect to $E_{\alpha_2}(t^{\alpha_2})$.

Proof. From Theorem 3.5, the system (39) has a unique solution $(p(t), q(t)) \in X \times X$. Next, we derive a representation of the solution. Subtracting the first expression from the second in (39), we get

$${}^H D_{0^+}^{\alpha_2,\beta_2}\big[{}^H D_{0^+}^{\alpha_1,\beta_1}+\delta_1(t)\big]q(t) = {}^H D_{0^+}^{\alpha_2,\beta_2}\big[{}^H D_{0^+}^{\alpha_1,\beta_1}+\lambda_1(t)\big]p(t) - (h_1(t)-h_2(t)). \tag{41}$$

Applying the operator $I_{0^+}^{\alpha_2}$ to (41) and since $[{}^H D_{0^+}^{\alpha_1,\beta_1}+\delta_1(t)]q(t) \in C_{1-\gamma_1}$ and $[{}^H D_{0^+}^{\alpha_1,\beta_1}+\lambda_1(t)]p(t) \in C_{1-\gamma_1}$, we have

$$\begin{aligned} I_{0^+}^{1-\gamma_2}\Big\{\big[{}^H D_{0^+}^{\alpha_1,\beta_1}+\delta_1(s)\big]q(s)\Big\}(0^+) &= 0, \\ I_{0^+}^{1-\gamma_2}\Big\{\big[{}^H D_{0^+}^{\alpha_1,\beta_1}+\lambda_1(s)\big]p(s)\Big\}(0^+) &= 0, \end{aligned}$$

and

$$\big[{}^H D_{0^+}^{\alpha_1,\beta_1}+\delta_1(t)\big]q(t) = \big[{}^H D_{0^+}^{\alpha_1,\beta_1}+\lambda_1(t)\big]p(t) - I_{0^+}^{\alpha_2}(h_1(t)-h_2(t)).$$

Multiplication of both sides of the above equation by $\lambda_2(t)$, in conjunction with the first equation in (39), leads to

$$\big[{}^H D_{0^+}^{\alpha_2,\beta_2}+\delta_2(t)\big]\big[{}^H D_{0^+}^{\alpha_1,\beta_1}+\lambda_1(t)\big]p(t)+\lambda_2(t)\Big\{\big[{}^H D_{0^+}^{\alpha_1,\beta_1}+\lambda_1(t)\big]p(t)-I_{0^+}^{\alpha_2}(h_1(t)-h_2(t))\Big\} = h_1(t),$$

that is

$$\big[{}^H D_{0^+}^{\alpha_2,\beta_2}+\delta_2(t)+\lambda_2(t)\big]\big[{}^H D_{0^+}^{\alpha_1,\beta_1}+\lambda_1(t)\big]p(t) = \lambda_2(t)I_{0^+}^{\alpha_2}[h_1(t)-h_2(t)]+h_1(t).$$

By Theorem 3.4, $p(t)$ can be represented by the first expression in (40). Similarly, we obtain

$$\big[{}^H D_{0^+}^{\alpha_2,\beta_2}+\lambda_2(t)+\delta_2(t)\big]\big[{}^H D_{0^+}^{\alpha_1,\beta_1}+\delta_1(t)\big]q(t) = -\delta_2(t)I_{0^+}^{\alpha_2}[h_1(t)-h_2(t)]+h_2(t),$$

and $q(t)$ can be represented by the second expression in (40). □

5.3. **Case III.** Let $\lambda_1(t) = \delta_1(t) = 0$ in (2) and

$$\begin{aligned} f_1(t,p(t),{}^H D_{0^+}^{\alpha_1,\beta_1}p(t),q(t),{}^H D_{0^+}^{\alpha_1,\beta_1}q(t)) &:= -a_{11}\,{}^H D_{0^+}^{\alpha_1,\beta_1}p(t) - a_{12}p(t) - a_{13}\,{}^H D_{0^+}^{\alpha_1,\beta_1}q(t) + r_1(t), \\ f_2(t,p(t),{}^H D_{0^+}^{\alpha_1,\beta_1}p(t),q(t),{}^H D_{0^+}^{\alpha_1,\beta_1}q(t)) &:= -a_{21}\,{}^H D_{0^+}^{\alpha_1,\beta_1}q(t) - a_{22}q(t) - a_{23}\,{}^H D_{0^+}^{\alpha_1,\beta_1}p(t) + r_2(t), \end{aligned}$$

then the system (2) reduces to the following system

$$\begin{cases} \big[{}^H D_{0^+}^{\alpha_2,\beta_2}+a_{11}\big]\,{}^H D_{0^+}^{\alpha_1,\beta_1}p(t) + a_{12}p(t) + a_{13}\,{}^H D_{0^+}^{\alpha_1,\beta_1}q(t) = r_1(t),\ t \in J, \\ \big[{}^H D_{0^+}^{\alpha_2,\beta_2}+a_{21}\big]\,{}^H D_{0^+}^{\alpha_1,\beta_1}q(t) + a_{22}q(t) + a_{23}\,{}^H D_{0^+}^{\alpha_1,\beta_1}p(t) = r_2(t),\ t \in J, \\ (I_{0^+}^{1-\gamma_1}p)(0^+) = p_0, \quad (I_{0^+}^{1-\gamma_1}q)(0^+) = q_0. \end{cases} \tag{42}$$

Theorem 5.3. *Let* $r_i(t) \in C_{1-\gamma_1} (i = 1, 2)$, *if*

$$5\max_{\substack{1\le i\le 2\\ 1\le j\le 3}}\{|a_{ij}|\}\Gamma(\gamma_1)\max\Big\{\frac{T^{\alpha_1}}{\Gamma(\alpha_1+\gamma_1)}, \frac{T^{\alpha_2}}{\Gamma(\alpha_2+\gamma_1)}, \frac{T^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+\gamma_1)}\Big\} < 1,$$

then the system (42) has a unique solution and the system (42) is UH stable and UHML stable with respect to $E_{\alpha_2}(t^{\alpha_2})$.

Proof. By Theorem 3.5, the system (42) has a unique solution given by:

$$\begin{cases} p(t) = I_{0+}^{\alpha_1+\alpha_2}[-a_{11}\,{}^{H}D_{0+}^{\alpha_1,\beta_1}p(t) - a_{12}p(t) - a_{13}\,{}^{H}D_{0+}^{\alpha_1,\beta_1}q(t) + r_1(t)] + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0,\ t \in J, \\ q(t) = I_{0+}^{\alpha_1+\alpha_2}[-a_{21}\,{}^{H}D_{0+}^{\alpha_1,\beta_1}q(t) - a_{22}q(t) - a_{23}\,{}^{H}D_{0+}^{\alpha_1,\beta_1}p(t) + r_2(t)] + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}q_0,\ t \in J. \end{cases}$$

From Theorem 2.1, by simplifying the terms on the right-hand sides of the above equations, we derive

$$\begin{cases} p(t) = -I_{0+}^{\alpha_2}[a_{11} + a_{12}I_{0+}^{\alpha_1}]p(t) - a_{13}I_{0+}^{\alpha_2}q(t) + I_{0+}^{\alpha_1+\alpha_2}r_1(t) + \frac{t^{\gamma_1+\alpha_2-1}}{\Gamma(\gamma_1+\alpha_2)}(a_{11}p_0 + a_{13}q_0) + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}p_0,\ t \in J, \\ q(t) = -I_{0+}^{\alpha_2}[a_{21} + a_{22}I_{0+}^{\alpha_1}]q(t) - a_{23}I_{0+}^{\alpha_2}p(t) + I_{0+}^{\alpha_1+\alpha_2}r_2(t) + \frac{t^{\gamma_1+\alpha_2-1}}{\Gamma(\gamma_1+\alpha_2)}(a_{21}q_0 + a_{23}p_0) + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}q_0,\ t \in J. \end{cases}$$

By Theorem 4.1 and Theorem 4.2, the system (42) is UH stable and UHML stable with respect to $E_{\alpha_2}(t^{\alpha_2})$. □

6. Applications

This section provides three illustrative examples.

Example 6.1. We consider the system of Langevin FDEs with variable coefficients

$$\begin{cases} \big[{}^{H}D_{0+}^{\frac14,\frac15} + \frac{t^2}{15}\big]\big[{}^{H}D_{0+}^{\frac35,\frac13} + t^{\frac12}\big]p(t) + \frac{t^2}{10}\big[{}^{H}D_{0+}^{\frac35,\frac13} + t^{\frac12}\big]q(t) = t^{-\frac13},\ t \in (0,1], \\ \big[{}^{H}D_{0+}^{\frac14,\frac15} + \frac{t^2}{10}\big]\big[{}^{H}D_{0+}^{\frac35,\frac13} + t^{\frac12}\big]q(t) + \frac{t^2}{15}\big[{}^{H}D_{0+}^{\frac35,\frac13} + t^{\frac12}\big]p(t) = t^{-\frac13} - t^{-\frac15},\ t \in (0,1], \\ (I_{0+}^{\frac{4}{15}}p)(0^+) = 1, \quad (I_{0+}^{\frac{4}{15}}q)(0^+) = 2. \end{cases} \tag{43}$$

Set $\alpha_1 = \frac35, \beta_1 = \frac13$, $\alpha_2 = \frac14, \beta_2 = \frac15$, $\lambda_1(t) = t^{\frac12}$, $\lambda_2(t) = \frac{t^2}{10}$, $\delta_1(t) = t^{\frac12}$, $\delta_2(t) = \frac{t^2}{15}$, $h_1(t) = t^{-\frac13}$ and $h_2(t) = t^{-\frac13} - t^{-\frac15}$, we can rewrite (43) to (39) and obtain $\gamma_1 = \frac{11}{15}$, $M^{\diamond} = \frac{1}{10}$ and

$$\begin{aligned} \overline{M} &:= \frac12\Gamma(\gamma_1)\max\{\frac{T^{\alpha_1}}{\Gamma(\alpha_1+\gamma_1)}, \frac{T^{\alpha_2}}{\Gamma(\alpha_2+\gamma_1)}, \frac{T^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+\gamma_1)}\} \\ &= \frac12\Gamma(\frac{11}{15})\max\{\frac{1}{\Gamma(\frac43)}, \frac{1}{\Gamma(\frac{59}{60})}, \frac{1}{\Gamma(\frac{19}{12})}\} \approx 0.70 < 1. \end{aligned}$$

From Theorem 5.2, it follows that the solution $(p(t), q(t))$ can be represented by (40), that is

$$\begin{cases} p(t) = \sum\limits_{k=0}^{\infty}(-I_{0+}^{\frac35}t^{\frac12})^k I_{0+}^{\frac35}\sum\limits_{n=0}^{\infty}(-1)^n\big(I_{0+}^{\frac14}\frac{t^2}{6}\big)^n I_{0+}^{\frac14}\big[\frac{t^2}{10}I_{0+}^{\frac14}t^{-\frac15} + t^{-\frac13}\big] + \sum\limits_{k=0}^{\infty}(-I_{0+}^{\frac35}t^{\frac12})^k\frac{t^{-\frac{4}{15}}}{\Gamma(\frac{11}{15})}, \\ \\ q(t) = \sum\limits_{k=0}^{\infty}(-I_{0+}^{\frac35}t^{\frac12})^k I_{0+}^{\frac35}\sum\limits_{n=0}^{\infty}(-1)^n\big(I_{0+}^{\frac14}\frac{t^2}{6}\big)^n I_{0+}^{\frac14}\big[-\frac{t^2}{15}I_{0+}^{\frac14}t^{-\frac15} + t^{-\frac13} - t^{-\frac15}\big] + 2\sum\limits_{k=0}^{\infty}(-I_{0+}^{\frac35}t^{\frac12})^k\frac{t^{-\frac{4}{15}}}{\Gamma(\frac{11}{15})}. \end{cases}$$

It can be readily observed that the following assertion holds:

$$(t^{\tau_1}I_{0+}^{\omega})^n t^{\tau_2-1} = \prod_{i=0}^{n-1}\frac{\Gamma(i(\omega+\tau_1)+\tau_2)}{\Gamma(i(\omega+\tau_1)+\tau_2+\omega)}t^{n(\omega+\tau_1)+\tau_2-1},\ \tau_1, \tau_2, \omega > 0,\ n \geq 1.$$

Then, we deduce

$$
\begin{aligned}
& I_{0^+}^{\omega}\sum_{n=0}^{\infty}(-1)^n(t^{\tau_1}I_{0^+}^{\omega})^n t^{\tau_2-1}\\
&= I_{0^+}^{\omega}t^{\tau_2-1}+I_{0^+}^{\omega}\sum_{n=1}^{\infty}(-1)^n(t^{\tau_1}I_{0^+}^{\omega})^n t^{\tau_2-1}\\
&= \frac{\Gamma(\tau_2)}{\Gamma(\omega+\tau_2)}t^{\omega+\tau_2-1}+I_{0^+}^{\omega}\sum_{n=1}^{\infty}(-1)^n\prod_{i=0}^{n-1}\frac{\Gamma(i(\omega+\tau_1)+\tau_2)}{\Gamma(i(\omega+\tau_1)+\omega+\tau_2)}t^{n(\omega+\tau_1)+\tau_2-1}\\
&= \sum_{n=0}^{\infty}(-1)^n\prod_{i=0}^{n}\frac{\Gamma(i(\omega+\tau_1)+\tau_2)}{\Gamma(i(\omega+\tau_1)+\omega+\tau_2)}t^{n(\omega+\tau_1)+\omega+\tau_2-1}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \sum_{k=0}^{\infty}(-I_{0^+}^{\frac{3}{5}}t^{\frac{1}{2}})^k I_{0^+}^{\frac{3}{5}}\sum_{n=0}^{\infty}(-1)^n\big(I_{0^+}^{\frac{1}{4}}\frac{t^2}{6}\big)^n I_{0^+}^{\frac{1}{4}}t^{\theta-1}\\
={}& I_{0^+}^{\frac{3}{5}}\sum_{k=0}^{\infty}(-t^{\frac{1}{2}}I_{0^+}^{\frac{3}{5}})^k I_{0^+}^{\frac{1}{4}}\sum_{n=0}^{\infty}(-\frac{1}{6})^n\big(t^2 I_{0^+}^{\frac{1}{4}}\big)^n t^{\theta-1}\\
={}& I_{0^+}^{\frac{3}{5}}\sum_{k=0}^{\infty}(-t^{\frac{1}{2}}I_{0^+}^{\frac{3}{5}})^k\sum_{n=0}^{\infty}(-\frac{1}{6})^n\prod_{i=0}^{n}\frac{\Gamma(\frac{9}{4}i+\theta)}{\Gamma(\frac{9}{4}i+\frac{1}{4}+\theta)}t^{\frac{9}{4}n-\frac{3}{4}+\theta}\\
={}& \sum_{k=0}^{\infty}(-1)^k\sum_{n=0}^{\infty}(-\frac{1}{6})^n\prod_{i=0}^{n}\frac{\Gamma(\frac{9}{4}i+\theta)}{\Gamma(\frac{9}{4}i+\frac{1}{4}+\theta)}\prod_{j=0}^{k}\frac{\Gamma(\frac{11}{10}j+\frac{9}{4}n+\frac{1}{4}+\theta)}{\Gamma(\frac{11}{10}j+\frac{9}{4}n+\frac{17}{20}+\theta)}t^{\frac{11}{10}k+\frac{9}{4}n-\frac{3}{20}+\theta}. \qquad (44)
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
p(t) ={}& \frac{1}{10}\frac{\Gamma(\frac{4}{5})}{\Gamma(\frac{21}{20})}\sum_{k=0}^{\infty}(-1)^k\sum_{n=0}^{\infty}(-\frac{1}{6})^n\prod_{i=0}^{n}\frac{\Gamma(\frac{9}{4}i+\frac{61}{20})}{\Gamma(\frac{9}{4}i+\frac{33}{10})}\prod_{j=0}^{k}\frac{\Gamma(\frac{11}{10}j+\frac{9}{4}n+\frac{33}{10})}{\Gamma(\frac{11}{10}j+\frac{9}{4}n+\frac{39}{10})}t^{\frac{11}{10}k+\frac{9}{4}n+\frac{29}{10}}\\
&+\sum_{k=0}^{\infty}(-1)^k\sum_{n=0}^{\infty}(-\frac{1}{6})^n\prod_{i=0}^{n}\frac{\Gamma(\frac{9}{4}i+\frac{2}{3})}{\Gamma(\frac{9}{4}i+\frac{11}{12})}\prod_{j=0}^{k}\frac{\Gamma(\frac{11}{10}j+\frac{9}{4}n+\frac{11}{12})}{\Gamma(\frac{11}{10}j+\frac{9}{4}n+\frac{91}{60})}t^{\frac{11}{10}k+\frac{9}{4}n+\frac{31}{60}}\\
&+\frac{1}{\Gamma(\frac{2}{15})}\sum_{k=0}^{\infty}(-1)^k\prod_{j=0}^{k}\frac{\Gamma(\frac{11}{10}j+\frac{2}{15})}{\Gamma(\frac{11}{10}j+\frac{11}{15})}t^{\frac{11}{10}k-\frac{4}{15}}. \qquad (45)
\end{aligned}
$$

In the same way, from (44), we get

$$
\begin{aligned}
q(t) ={}& -\frac{1}{15}\frac{\Gamma(\frac{4}{5})}{\Gamma(\frac{21}{20})}\sum_{k=0}^{\infty}(-1)^k\sum_{n=0}^{\infty}(-\frac{1}{6})^n\prod_{i=0}^{n}\frac{\Gamma(\frac{9}{4}i+\frac{61}{20})}{\Gamma(\frac{9}{4}i+\frac{33}{10})}\prod_{j=0}^{k}\frac{\Gamma(\frac{11}{10}j+\frac{9}{4}n+\frac{33}{10})}{\Gamma(\frac{11}{10}j+\frac{9}{4}n+\frac{39}{10})}t^{\frac{11}{10}k+\frac{9}{4}n+\frac{29}{10}}\\
&+\sum_{k=0}^{\infty}(-1)^k\sum_{n=0}^{\infty}(-\frac{1}{6})^n\prod_{i=0}^{n}\frac{\Gamma(\frac{9}{4}i+\frac{2}{3})}{\Gamma(\frac{9}{4}i+\frac{11}{12})}\prod_{j=0}^{k}\frac{\Gamma(\frac{11}{10}j+\frac{9}{4}n+\frac{11}{12})}{\Gamma(\frac{11}{10}j+\frac{9}{4}n+\frac{91}{60})}t^{\frac{11}{10}k+\frac{9}{4}n+\frac{31}{60}}\\
&-\sum_{k=0}^{\infty}(-1)^k\sum_{n=0}^{\infty}(-\frac{1}{6})^n\prod_{i=0}^{n}\frac{\Gamma(\frac{9}{4}i+\frac{4}{5})}{\Gamma(\frac{9}{4}i+\frac{21}{20})}\prod_{j=0}^{k}\frac{\Gamma(\frac{11}{10}j+\frac{9}{4}n+\frac{21}{20})}{\Gamma(\frac{11}{10}j+\frac{9}{4}n+\frac{33}{20})}t^{\frac{11}{10}k+\frac{9}{4}n+\frac{13}{20}}\\
&+\frac{2}{\Gamma(\frac{2}{15})}\sum_{k=0}^{\infty}(-1)^k\prod_{j=0}^{k}\frac{\Gamma(\frac{11}{10}j+\frac{2}{15})}{\Gamma(\frac{11}{10}j+\frac{11}{15})}t^{\frac{11}{10}k-\frac{4}{15}}. \qquad (46)
\end{aligned}
$$

By Theorem 4.1 and Theorem 4.2, the system (43) is UH stable and UHML stable with respect to $E_{\frac{1}{4}}(t^{\frac{1}{4}})$.

Similarly, for $|\omega_i(t)| \le \varepsilon\,(i = 1, 2)$, the system

$$\begin{cases} \big[{}^{H}D_{0+}^{\frac{1}{4},\frac{1}{5}} + \frac{t^2}{15}\big]\big[{}^{H}D_{0+}^{\frac{3}{5},\frac{1}{3}} + t^{\frac{1}{2}}\big]p(t) + \frac{t^2}{10}\big[{}^{H}D_{0+}^{\frac{3}{5},\frac{1}{3}} + t^{\frac{1}{2}}\big]q(t) = t^{-\frac{1}{3}} + \omega_1(t),\ t \in (0,1], \\ \big[{}^{H}D_{0+}^{\frac{1}{4},\frac{1}{5}} + \frac{t^2}{10}\big]\big[{}^{H}D_{0+}^{\frac{3}{5},\frac{1}{3}} + t^{\frac{1}{2}}\big]q(t) + \frac{t^2}{15}\big[{}^{H}D_{0+}^{\frac{3}{5},\frac{1}{3}} + t^{\frac{1}{2}}\big]p(t) = t^{-\frac{1}{3}} - t^{-\frac{1}{5}} + \omega_2(t),\ t \in (0,1], \\ (I_{0+}^{\frac{4}{15}}p)(0^+) = 1, \quad (I_{0+}^{\frac{4}{15}}q)(0^+) = 2, \end{cases} \tag{47}$$

has a unique solution $(\widetilde{p}(t), \widetilde{q}(t))$ given by

$$\begin{cases} \widetilde{p}(t) = \sum\limits_{k=0}^{\infty}(-I_{0+}^{\frac{3}{5}}t^{\frac{1}{2}})^k I_{0+}^{\frac{3}{5}} \sum\limits_{n=0}^{\infty}(-1)^n\big(I_{0+}^{\frac{1}{4}}\frac{t^2}{6}\big)^n I_{0+}^{\frac{1}{4}}\big[\frac{t^2}{10}I_{0+}^{\frac{1}{4}}t^{-\frac{1}{5}} + t^{-\frac{1}{3}} + \omega_1(t)\big] + \sum\limits_{k=0}^{\infty}(-I_{0+}^{\frac{3}{5}}t^{\frac{1}{2}})^k \frac{t^{-\frac{4}{15}}}{\Gamma(\frac{11}{15})}, \\ \widetilde{q}(t) = \sum\limits_{k=0}^{\infty}(-I_{0+}^{\frac{3}{5}}t^{\frac{1}{2}})^k I_{0+}^{\frac{3}{5}} \sum\limits_{n=0}^{\infty}(-1)^n\big(I_{0+}^{\frac{1}{4}}\frac{t^2}{6}\big)^n I_{0+}^{\frac{1}{4}}\big[-\frac{t^2}{15}I_{0+}^{\frac{1}{4}}t^{-\frac{1}{5}} + t^{-\frac{1}{3}} - t^{-\frac{1}{5}} + \omega_2(t)\big] \\ +2\sum\limits_{k=0}^{\infty}(-I_{0+}^{\frac{3}{5}}t^{\frac{1}{2}})^k \frac{t^{-\frac{4}{15}}}{\Gamma(\frac{11}{15})}\Gamma(\frac{11}{15}). \end{cases} \tag{48}$$

Below, we plot two graphs of the approximate solutions for the systems (43) and (47). For this purpose, we set $\omega_i(t) = \varepsilon\,(i = 1, 2)$ and denote the partial sums of the aforementioned series (45), (46) and (48) for k ranging from 0 to K and n ranging from 0 to N as $p_{K,N}(t)$, $q_{K,N}(t)$, $p_{K,N,\varepsilon}(t)$, and $q_{K,N,\varepsilon}(t)$, respectively.

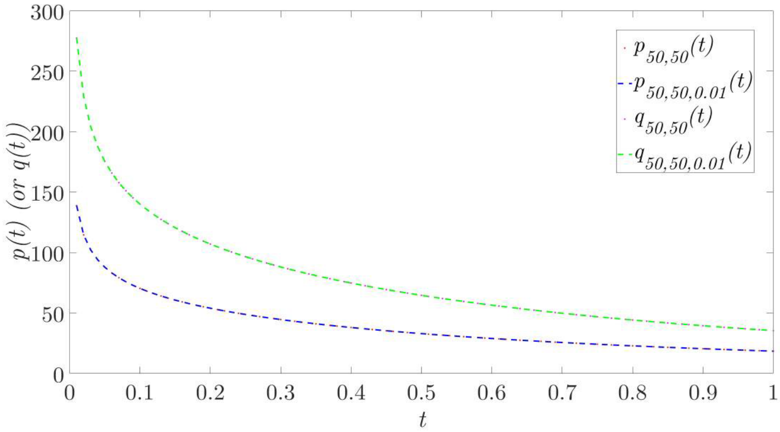

Figure 1. The graph of $p_{50,50}(t)$, $p_{50,50,0.01}(t)$, $q_{50,50}(t)$ and $q_{50,50,0.01}(t)$.

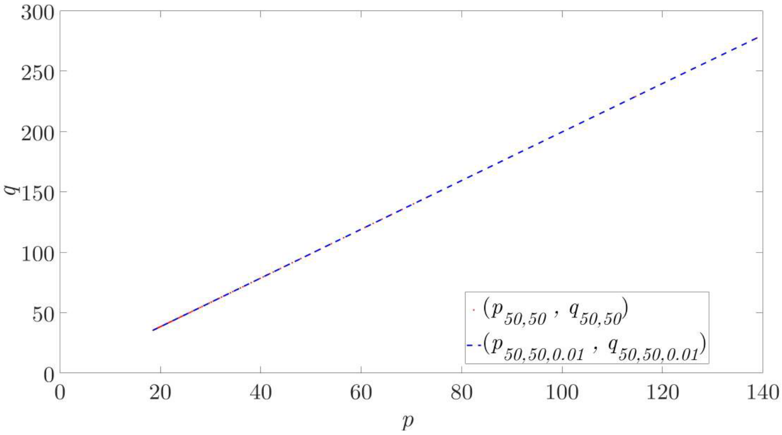

Figure 2. The graph of UH stability of the system (43).

Example 6.2. We consider the system of Langevin FDEs with Hilfer fractional derivatives

$$\begin{cases} \big[{}^{H}D_{0+}^{\frac{1}{4},\frac{1}{6}} + \frac{1}{10}\big]\big[{}^{H}D_{0+}^{\frac{1}{2},\frac{1}{5}} + \frac{1}{12}\big]p(t) = \frac{t}{10}q(t) + t^{-\frac{4}{5}}E_{\frac{1}{4},\frac{1}{5}}(-\frac{1}{5}t^{\frac{1}{4}}),\ t \in (0,1], \\ \big[{}^{H}D_{0+}^{\frac{1}{4},\frac{1}{6}} + \frac{1}{16}\big]\big[{}^{H}D_{0+}^{\frac{1}{2},\frac{1}{5}} + \frac{1}{11}\big]q(t) = \frac{t^2}{15}p(t) + t^{-\frac{2}{3}}E_{\frac{1}{4},\frac{1}{3}}(-\frac{1}{8}t^{\frac{1}{4}}),\ t \in (0,1], \\ (I_{0+}^{\frac{2}{5}}p)(0^+) = 0, \quad (I_{0+}^{\frac{2}{5}}q)(0^+) = 0. \end{cases} \tag{49}$$

When we set $\alpha_1 = \frac{1}{2}, \beta_1 = \frac{1}{5}, \alpha_2 = \frac{1}{4}, \beta_2 = \frac{1}{6}$, $\lambda_1 = \frac{1}{12}$, $\lambda_2 = \frac{1}{10}$, $\delta_1 = \frac{1}{11}$, $\delta_2 = \frac{1}{16}$, $g_1(t, q(t)) = \frac{t}{10}q(t) + t^{-\frac{4}{5}}E_{\frac{1}{4},\frac{1}{5}}(-\frac{1}{5}t^{\frac{1}{4}})$ and $g_2(t, p(t)) = \frac{t^2}{15}p(t) + t^{-\frac{2}{3}}E_{\frac{1}{4},\frac{1}{3}}(-\frac{1}{8}t^{\frac{1}{4}})$, we are able to rewrite (49) as (33) and find that $\gamma_1 = \frac{3}{5}$, $M^* = \frac{1}{10}$ and

$$\begin{aligned} \widetilde{M} \ &:= \ 5M^*\Gamma(\gamma_1)\max\{\frac{T^{\alpha_1}}{\Gamma(\alpha_1+\gamma_1)}, \frac{T^{\alpha_2}}{\Gamma(\alpha_2+\gamma_1)}, \frac{T^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+\gamma_1)}\} \\ &= \ \frac{1}{2}\Gamma(\frac{3}{5})\max\{\frac{1}{\Gamma(\frac{11}{10})}, \frac{1}{\Gamma(\frac{17}{20})}, \frac{1}{\Gamma(\frac{27}{20})}\} \approx 0.84 < 1. \end{aligned}$$

From Corollary 5.1 and Lemma 2.4, it follows that the solution $(p(t), q(t))$ can be represented by (34), that is

$$\begin{cases} p(t) = \int\limits_0^t (t-s)^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}(-\frac{1}{12}(t-s)^{\frac{1}{2}})\Big\{\int\limits_0^s (s-\tau)^{-\frac{3}{4}}E_{\frac{1}{4},\frac{1}{4}}(-\frac{1}{10}(s-\tau)^{\frac{1}{4}})[\frac{\tau}{10}q(\tau) + \tau^{-\frac{4}{5}}E_{\frac{1}{4},\frac{1}{5}}(-\frac{1}{5}\tau^{\frac{1}{4}})]d\tau\Big\}ds \\ \quad = \frac{1}{10}\int\limits_0^t (t-s)^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}(-\frac{1}{12}(t-s)^{\frac{1}{2}})\int\limits_0^s (s-\tau)^{-\frac{3}{4}}E_{\frac{1}{4},\frac{1}{4}}(-\frac{1}{10}(s-\tau)^{\frac{1}{4}})\tau q(\tau)d\tau ds \\ \quad + \int\limits_0^t (t-s)^{-\frac{3}{4}}E_{\frac{1}{2},\frac{1}{2}}(-\frac{1}{12}(t-s)^{\frac{1}{2}})s^{-\frac{11}{20}}\big[2E_{\frac{1}{4},\frac{9}{20}}(-\frac{1}{5}s^{\frac{1}{4}}) - E_{\frac{1}{4},\frac{9}{20}}(-\frac{1}{10}s^{\frac{1}{4}})\big]ds, \\ q(t) = \int\limits_0^t (t-s)^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}(-\frac{1}{11}(t-s)^{\frac{1}{2}})\{\int\limits_0^s (s-\tau)^{-\frac{3}{4}}E_{\frac{1}{4},\frac{1}{4}}(-\frac{1}{16}(s-\tau)^{\frac{1}{4}})\big[\frac{\tau^2}{15}p(\tau) + \tau^{-\frac{2}{3}}E_{\frac{1}{4},\frac{1}{3}}(-\frac{1}{8}\tau^{\frac{1}{4}})\big]d\tau\}ds \\ \quad = \frac{1}{15}\int\limits_0^t (t-s)^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}(-\frac{1}{11}(t-s)^{\frac{1}{2}})\int\limits_0^s (s-\tau)^{-\frac{3}{4}}E_{\frac{1}{4},\frac{1}{4}}(-\frac{1}{16}(s-\tau)^{\frac{1}{4}})\tau^2 p(\tau)d\tau ds \\ \quad + \int\limits_0^t (t-s)^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}(-\frac{1}{11}(t-s)^{\frac{1}{2}})s^{-\frac{5}{12}}\big[2E_{\frac{1}{4},\frac{7}{12}}(-\frac{1}{8}s^{\frac{1}{4}}) - E_{\frac{1}{4},\frac{7}{12}}(-\frac{1}{16}s^{\frac{1}{4}})\big]ds. \end{cases}$$

An RLC circuit is an electrical system consisting of resistors (R), inductors (L), and capacitors (C) and source voltages (e) connected in series or parallel, exhibiting oscillatory behavior and resonant frequency characteristics. The classical circuit theory paradigm inherently idealized inductors and capacitors as integer-order components, mathematically representing their voltage-current dynamics via first-order differential or integral equations. However, in engineering applications, inductors and capacitors exhibit fractional-order properties, which cannot be accounted for by the concepts of integer-order circuits. Analyzing these components and their circuits using fractional calculus can reveal new characteristics that are not observable in integer-order systems. A comparison between fractional-order and integer-order RLC systems is presented in Table 1.

Table 1. Comparison between Integer-Order and Fractional-Order RLC Circuits

Aspect	**Integer-Order RLC Circuit**	**Fractional-Order RLC Circuit**
Physical Basis	An ideal model is characterized by separate and instantaneous energy storage (L, C) and dissipation (R).	The model is more physically realistic as it incorporates memory effects and distributed characteristics.
Component Modeling	**Ideal Elements:** • C: $i_C(t) = C\frac{du_C(t)}{dt}$ • L: $u_L(t) = L\frac{di_L(t)}{dt}$	**Fractional (Non-Ideal) Elements:** • Fractional Capacitor: $i_C(t) = CD_t^{\alpha_1}u_C(t)(0 < \alpha_1 < 1)$ • Fractional Inductor: $u_L(t) = LD_t^{\alpha_2}i_L(t)(0 < \alpha_2 < 1)$
Transient Response	Fixed responses Decay is described by exponential functions.	More complex responses Decay is described by special functions like the Mittag-Leffler function, exhibiting non-exponential properties.
Applications	Used for analyzing circuits composed of ideal components	Used for designing devices with specific frequency responses

For detailed information about RLC circuits, we refer to [18, 44].

Next, we apply our result to solve a fractional-order RLC circuit system.

Example 6.3. We consider the linear electrical circuit shown on Figure 3 with resistances $R_j (j = 1, 2, 3)$, capacitances $C_j (j = 1, 2)$, inductances $L_j (j = 1, 2)$ and source voltages $e_j(t) (j = 1, 2)$.

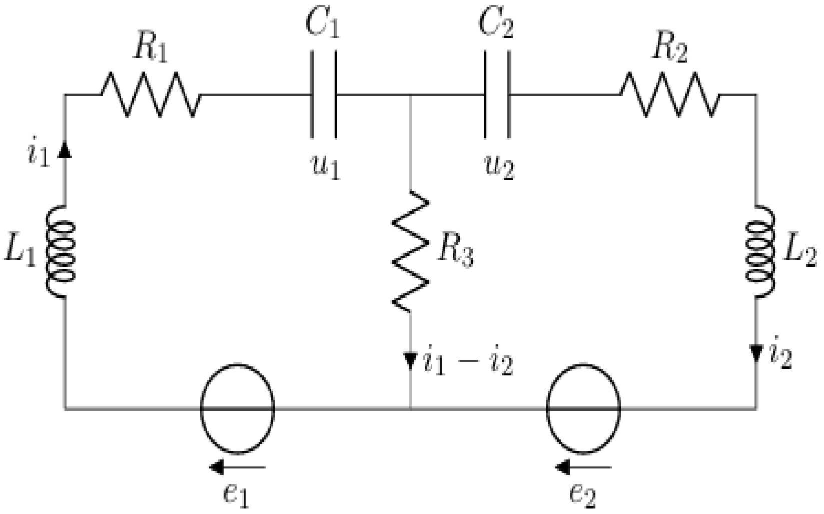

Figure 3. Electrical circuit of Example 6.3.

Applying Kirchhoff's laws, we write for the circuit the following equations:

$$i_1(t) = C_1 \, {}^{RL}D_{0^+}^{\alpha_1} u_1(t), \quad i_2(t) = C_2 \, {}^{RL}D_{0^+}^{\alpha_1} u_2(t), \tag{50}$$

$$e_1(t) = (R_1 + R_3)i_1(t) + L_1 \, {}^{RL}D_{0^+}^{\alpha_2} i_1(t) + u_1(t) - R_3 i_2(t), \tag{51}$$

$$e_2(t) = (R_2 + R_3)i_2(t) + L_2 \, {}^{RL}D_{0^+}^{\alpha_2} i_2(t) + u_2(t) - R_3 i_1(t), \tag{52}$$

where $0 < \alpha_2 < \alpha_1 < 1$, $u_1(t)$ and $u_2(t)$ are the voltages across the capacitors C_1 and C_2, respectively, $e_j(t) \in \mathcal{C} (j = 1, 2)$.

Furthermore, we can rewrite (50), (51) and (52) as the following system

$$\begin{cases} C_1 L_1 \, {}^{RL}D_{0^+}^{\alpha_2} \, {}^{RL}D_{0^+}^{\alpha_1} u_1(t) + C_1(R_1 + R_3) \, {}^{RL}D_{0^+}^{\alpha_1} u_1(t) + u_1(t) - C_2 R_3 \, {}^{RL}D_{0^+}^{\alpha_1} u_2(t) = e_1(t), \; t \in J, \\ C_2 L_2 \, {}^{RL}D_{0^+}^{\alpha_2} \, {}^{RL}D_{0^+}^{\alpha_1} u_2(t) + C_2(R_2 + R_3) \, {}^{RL}D_{0^+}^{\alpha_1} u_2(t) + u_2(t) - C_1 R_3 \, {}^{RL}D_{0^+}^{\alpha_1} u_1(t) = e_2(t), \; t \in J. \end{cases} \tag{53}$$

Next, we study (53) under initial value conditions

$$(I_{0^+}^{1-\alpha_1}u_1)(0^+) = c_0,\ (I_{0^+}^{1-\alpha_1}u_2)(0^+) = c_1.$$

Set

$$a_{11} = \frac{R_1+R_3}{L_1}, \quad a_{21} = \frac{R_2+R_3}{L_2}, \quad a_{12} = \frac{1}{L_1C_1}, \quad a_{22} = \frac{1}{L_2C_2},$$
$$a_{13} = -\frac{C_2R_3}{C_1L_1}, \quad a_{23} = -\frac{C_1R_3}{C_2L_2}, \quad r_1(t) = \frac{e_1(t)}{C_1L_1}, \quad r_2(t) = \frac{e_2(t)}{C_2L_2},$$

then system (53) can be transformed into the system (42)(when $\beta_i = 0(i=1,2)$). By Theorem 5.3, if

$$5\max_{\substack{1\le i\le 2\\ 1\le j\le 3}}\{|a_{ij}|\}\Gamma(\alpha_1)\max\left\{\frac{T^{\alpha_1}}{\Gamma(2\alpha_1)}, \frac{T^{\alpha_2}}{\Gamma(\alpha_1+\alpha_2)}, \frac{T^{\alpha_1+\alpha_2}}{\Gamma(2\alpha_1+\alpha_2)}\right\} < 1,$$

the system (53) has a unique solution

$$\left\{\begin{array}{l} u_1(t) = -I_{0^+}^{\alpha_2}[a_{11}+a_{12}I_{0^+}^{\alpha_1}]u_1(t) - a_{13}I_{0^+}^{\alpha_2}u_2(t) + \Upsilon_1(t),\ t\in J,\\ u_2(t) = -I_{0^+}^{\alpha_2}[a_{21}+a_{22}I_{0^+}^{\alpha_1}]u_2(t) - a_{23}I_{0^+}^{\alpha_2}u_1(t) + \Upsilon_2(t),\ t\in J, \end{array}\right. \tag{54}$$

where

$$\begin{aligned} \Upsilon_1(t) &:= I_{0^+}^{\alpha_1+\alpha_2}r_1(t) + \frac{t^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}(a_{11}c_0 + a_{13}c_1) + \frac{t^{\alpha_1-1}}{\Gamma(\alpha_1)}c_0,\\ \Upsilon_2(t) &:= I_{0^+}^{\alpha_1+\alpha_2}r_2(t) + \frac{t^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}(a_{21}c_1 + a_{23}c_0) + \frac{t^{\alpha_1-1}}{\Gamma(\alpha_1)}c_1. \end{aligned}$$

From the proof of [25, Theorem 3 and Theorem 4], we further obtain

$$u_1(t) = \sum_{k=0}^{\infty}(-1)^k(a_{11}I_{0^+}^{\alpha_2} + a_{12}I_{0^+}^{\alpha_1+\alpha_2})^k\Big[-a_{13}I_{0^+}^{\alpha_2}u_2(t) + \Upsilon_1(t)\Big],\ t\in J.$$

For any function $f(t)\in C_{1-\alpha_1}$, one can derive that

$$\begin{aligned} &\sum_{k=0}^{\infty}(-1)^k\,(a_{11}I_{0^+}^{\alpha_2} + a_{12}I_{0^+}^{\alpha_1+\alpha_2})^k f(t)\\ =\ &\sum_{k=0}^{\infty}(-1)^k\sum_{\substack{i+j=k\\ i,j\ge 0}}\binom{k}{i\ j}a_{11}^i a_{12}^j\, I_{0^+}^{\alpha_2 i + j(\alpha_1+\alpha_2)}f(t)\\ =\ &\int_0^t\sum_{k=0}^{\infty}(-1)^k\sum_{\substack{i+j=k\\ i,j\ge 0}}\binom{k}{i\ j}a_{11}^i a_{12}^j\frac{(t-s)^{\alpha_2 i + j(\alpha_1+\alpha_2)-1}}{\Gamma(\alpha_2 i + j(\alpha_1+\alpha_2))}f(s)ds\\ =\ &\int_0^t\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(i+j)!}{i!j!}\cdot(-a_{11})^i(-a_{12})^j\frac{(t-s)^{\alpha_2 i + j(\alpha_1+\alpha_2)-1}}{\Gamma(\alpha_2 i + j(\alpha_1+\alpha_2))}f(s)ds\\ =\ &\int_0^t\sum_{i=0}^{\infty}(-a_{11})^i(t-s)^{i\alpha_2-1}\sum_{j=0}^{\infty}\frac{(i+j)!\big(-a_{12}(t-s)^{(\alpha_1+\alpha_2)}\big)^j}{i!j!\Gamma(j(\alpha_1+\alpha_2)+i\alpha_2)}f(s)ds\\ =\ &\sum_{n=0}^{\infty}(-a_{11})^n\int_0^t s^{n\alpha_2-1}E_{\alpha_1+\alpha_2,n\alpha_2}^{n+1}(-a_{12}s^{\alpha_1+\alpha_2})f(t-s)ds\\ =\ &\sum_{n=0}^{\infty}(-a_{11})^n(\mathbf{E}_{\alpha_1+\alpha_2,n\alpha_2;a_{12}}^{n+1}f)(t). \end{aligned}$$

By [19, Lemma 4 and Theorem 6], we have

$$\sum_{n=0}^{\infty}\frac{(-a_{11})^n}{\Gamma(\alpha_1+\alpha_2)}(\mathbf{E}^{n+1}_{\alpha_1+\alpha_2,n\alpha_2;a_{12}}s^{\alpha_1+\alpha_2-1})(t)=\sum_{n=0}^{\infty}(-a_{11})^n t^{(n+1)\alpha_2+\alpha_1-1}E^{n+1}_{\alpha_1+\alpha_2,(n+1)\alpha_2+\alpha_1}(-a_{12}t^{\alpha_1+\alpha_2}), \tag{55}$$

$$\frac{1}{\Gamma(\alpha_1)}\sum_{n=0}^{\infty}(-a_{11})^n(\mathbf{E}^{n+1}_{\alpha_1+\alpha_2,n\alpha_2;a_{12}}s^{\alpha_1-1})(t)=\sum_{n=0}^{\infty}(-a_{11})^n t^{n\alpha_2+\alpha_1-1}E^{n+1}_{\alpha_1+\alpha_2,n\alpha_2+\alpha_1}(-a_{12}t^{\alpha_1+\alpha_2}), \tag{56}$$

$$\sum_{n=0}^{\infty}(-a_{11})^n(\mathbf{E}^{n+1}_{\alpha_1+\alpha_2,n\alpha_2;a_{12}}I^{\alpha_1+\alpha_2}_{0+}r_1)(t)=\sum_{n=0}^{\infty}(-a_{11})^n(\mathbf{E}^{n+1}_{\alpha_1+\alpha_2,(n+1)\alpha_2+\alpha_1;a_{12}}r_1)(t). \tag{57}$$

Hence

$$u_1(t)=-a_{13}\sum_{n=0}^{\infty}(-a_{11})^n(\mathbf{E}^{n+1}_{\alpha_1+\alpha_2,(n+1)\alpha_2;a_{12}}u_2)(t)+\Omega(t), \tag{58}$$

where

$$\begin{aligned}\Omega(t) \quad := \quad & \frac{1}{C_1L_1}\sum_{n=0}^{\infty}(-a_{11})^n(\mathbf{E}^{n+1}_{\alpha_1+\alpha_2,(n+1)\alpha_2+\alpha_1;a_{12}}e_1)(t)\\ & +(a_{11}c_0+a_{13}c_1)\sum_{n=0}^{\infty}(-a_{11})^n t^{(n+1)\alpha_2+\alpha_1-1}E^{n+1}_{\alpha_1+\alpha_2,(n+1)\alpha_2+\alpha_1}(-a_{12}t^{\alpha_1+\alpha_2})\\ & +c_0\sum_{n=0}^{\infty}(-a_{11})^n t^{n\alpha_2+\alpha_1-1}E^{n+1}_{\alpha_1+\alpha_2,n\alpha_2+\alpha_1}(-a_{12}t^{\alpha_1+\alpha_2}).\end{aligned} \tag{59}$$

Substituting (58) into the second equation of (54) yields

$$u_2(t)=-I^{\alpha_2}_{0+}[a_{21}+a_{22}I^{\alpha_1}_{0+}]u_2(t)+a_{13}a_{23}\sum_{n=0}^{\infty}(-a_{11})^n(\mathbf{E}^{n+1}_{\alpha_1+\alpha_2,(n+2)\alpha_2;a_{12}}u_2)(t)-a_{23}I^{\alpha_2}_{0+}\Omega(t)+\Upsilon_2(t).$$

The application of Picard's successive approximation method leads to

$$u_2(t)=\sum_{k=0}^{\infty}(-1)^k\Big[a_{21}I^{\alpha_2}_{0+}+a_{22}I^{\alpha_1+\alpha_2}_{0+}-a_{13}a_{23}\sum_{n=0}^{\infty}(-a_{11})^n\mathbf{E}^{n+1}_{\alpha_1+\alpha_2,(n+2)\alpha_2;a_{12}}\Big]^k\Big[-a_{23}I^{\alpha_2}_{0+}\Omega(t)+\Upsilon_2(t)\Big].$$

Substitution of the above expression into (58) gives $u_1(t)$.

7. CONCLUSIONS

The system (1) represents a sophisticated mathematical model encompassing multiple derivatives, serving as a foundational framework from which diverse specialized systems can be derived. Notable examples include fractional-order Langevin systems and fractional-order RLC circuits. This framework can generate numerous previously unexplored models, highlighting its novel contributions to the field. Nevertheless, investigating the nonlinear system (1) with variable coefficients introduces profound analytical difficulties, particularly given that essential properties of the corresponding linear systems have yet to be fully characterized. In this study, we establish the existence and uniqueness of solutions to the system (1), as well as its stability. As particular cases of system (1), we can derive: (i) The corresponding results for fractional-order Langevin systems, and (ii) a mathematical framework of an RLC circuit. The methodological framework developed in this study can be directly extended to address more complex systems. An important direction for future research lies in extending the present analysis to examine the system (1) under boundary conditions, where several fundamental questions remain unresolved.

8. ACKNOWLEDGEMENT

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Huiwen Wang - received his Ph.D. degree in mathematics from University of Science and Technology of China in 2003. His research focuses on stability theorem and fractional differential equations.

Fang Li - received her Ph.D. degree in mathematics from University of Science and Technology of China in 2002. Her research focuses on fractional differential equations, semigroup operators and functional equations.